The renormalization group equation viewed combinatorially

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Augmented generating functions

Take a combinatorial class $\mathcal{C}$. Build a generating function but keep the objects.

$$\sum_{c \in \mathcal{C}} c x^{|c|} \in \mathbb{Q}[\mathcal{C}][[x]]$$

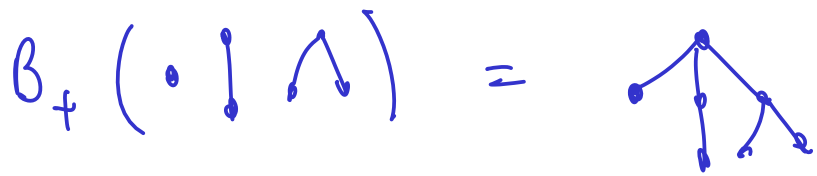
- Get the ordinary generating function by evaluating $c \mapsto 1$.
- Count with parameters by evaluating each object as a monomial in the parameters.
- More to today's point if $\mathcal{C}$ is a class of Feynman graphs (or rooted trees...) evaluate by **Feynman rules**.

Rooted trees

with no plane structure

Let $\mathcal{T}$ be a class of rooted trees. Identify forests of rooted trees with monomials in $\mathbb{Q}[\mathcal{T}]$.

Let $B_+(F)$ be the tree constructed by adding a new root above each tree from the forest F.
Eg:



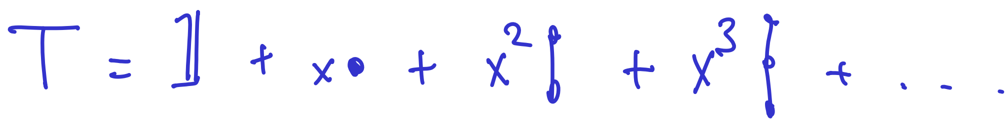
Tree recurrences

Let $T \in \mathbb{Q}[\mathcal{T}][[x]]$. What does

empty tree

$$T = \mathbb{I} + xB_+(T)$$

count?



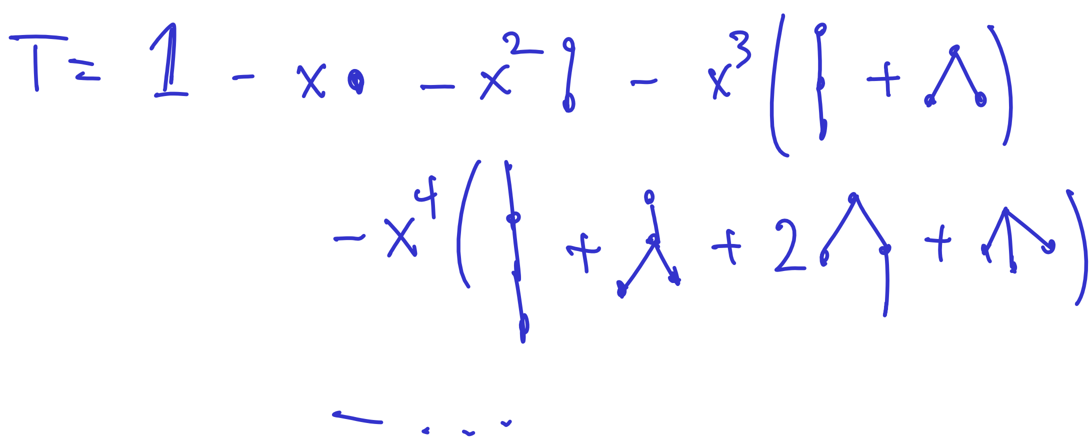
More tree recurrences

What does

$$T = \mathbb{I} - xB_+\left(\frac{1}{T}\right)$$

$T = \mathbb{1} - x\,\mathrm{Seq}(T)$

count?



Green functions

Think of rooted trees as representing the subdivergence structure of Feynman diagrams.

For us, Feynman rules are an evaluation map ϕ, say

$$\phi : \mathcal{T} \to \mathbb{C}[L] \quad \text{(simplified case)}$$

The Green function is ϕ applied to the augmented generating function.

$$G(x, L) = \phi\left(\sum_{t \in \mathcal{T}} t \, x^{|t|}\right) = \sum_{t \in \mathcal{T}} \phi(t) \, x^{|t|}$$

The actual physical Feynman rules build an integral from the Feynman graph.

A particular case

Consider

$$T = \mathbb{I} - xB_+\left(\frac{1}{T}\right)$$

and evaluate with the physical ϕ. After some work this gives

$$G(x,L) = 1 - xG(x,\partial_{-\rho})^{-1}(e^{-L\rho}-1)F(\rho)\Big|_{\rho=0}$$

where

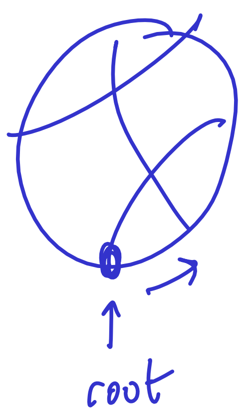
$$F(\rho) = \frac{f_0}{\rho} + f_1 + f_2\rho + f_3\rho^2 + \cdots$$

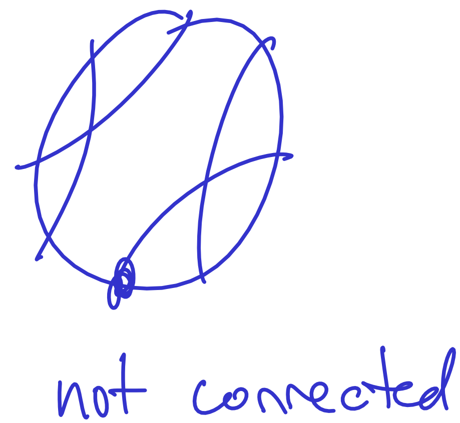
comes from the regularized Feynman integral for the primitive associated to •.

Rooted connected chord diagrams

Can solve this by a chord diagram expansion (with N. Marie, more general case with M. Hihn).

A chord diagram is *rooted* if it has a distinguished vertex.
A chord diagram is *connected* if no set of chords can be separated from the others by a line.
Eg:





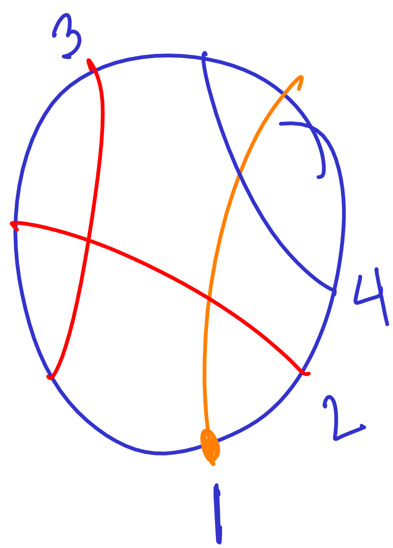
These are really just irreducible matchings of points along a line.

Recursive chord order

Let C be a connected rooted chord diagram. Order the chords recursively:

- c_1 is the root chord
- Order the connected components of $C \setminus c_1$ as they first appear running counterclockwise, $D_1, D_2, \ldots$. Recursively order the chords of D_1, then of D_2, and so on.

Eg:



terminal chords 3, 4

Terminal chords

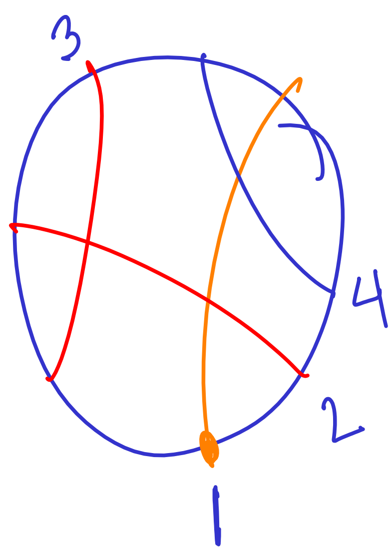
A chord is terminal if it only crosses chords which come before it in the recursive chord order. Let

$$t_1 < t_2 < \cdots < t_\ell$$

be the terminal chords of C. Then

- $b(C) = t_1$ and
- $f_C = f_{t_\ell - t_{\ell-1}} \cdots f_{t_3 - t_2} f_{t_2 - t_1} f_0^{|C|-\ell}$

Eg:



terminal chords 3, 4

$b(C) = 3$ $f_C = f_1 f_0^2$

Result

Theorem 1

$$G(x,L) = 1 - \sum_{i\geq 1} \frac{(-L)^i}{i!} \sum_{\substack{C \\ b(C)\geq i}} x^{|C|} f_C f_{b(C)-i}$$

solves

$$G(x,L) = 1 - xG(x,\partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)\Big|_{\rho=0}$$

where

$$F(\rho) = \frac{f_0}{\rho} + f_1 + f_2\rho + f_3\rho^2 + \cdots$$

The renormalization group equation

The **renormalization group equation** tells us how the coupling changes with the energy. It is very important physically.
For us it says

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} + \gamma(x)\right) G(x, L) = 0$$

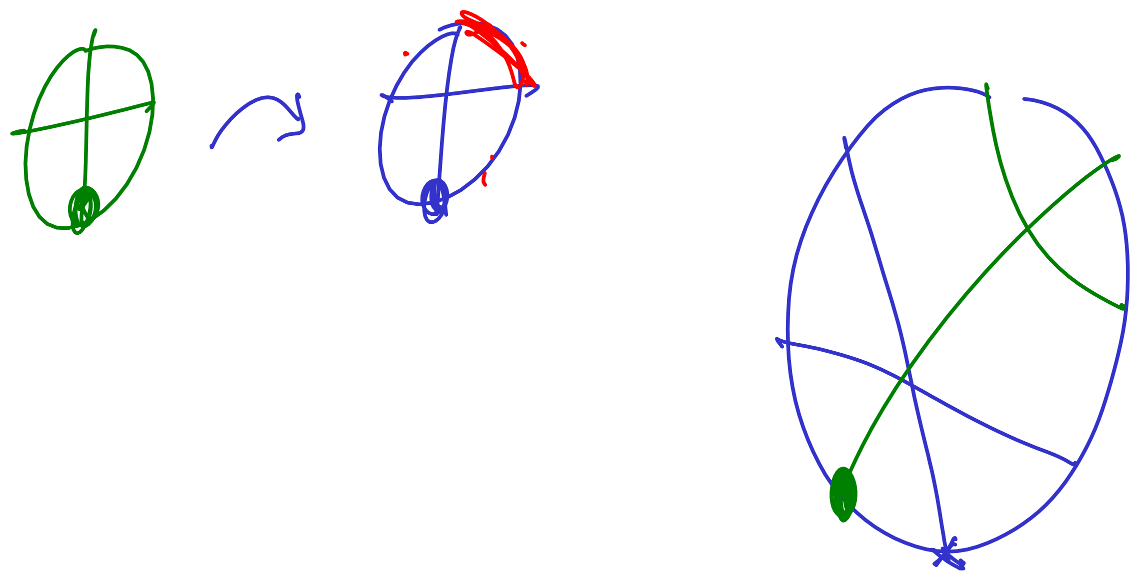
What happens if we apply it to the chord diagram expansion?

Chord diagram decomposition

We can insert a rooted connected chord diagram C_1 into another C_2, by

- choosing an interval of C_2 other than the one before the root
- putting the root of C_1 just before the root of C_2 and
- putting the rest of C_2 in the chosen interval

Eg:



Since the diagrams are connected C_1 and C_2 can be recovered.

A classical recurrence

This decomposition is classical. Nijenhuis and Wilf (1978) use it to prove the recurrence (originally due to Stein (1978) and rephrased by Riordan)

$$s_n = \sum_{k=1}^{n-1} (2k-1) s_k s_{n-k} \qquad \text{for } n \geq 2$$

where s_n is the number of connected rooted chord diagrams with n chords.

The recurrence translated

This recurrence can be extended to keep track of the terminal chords. Let

$$g_{k,i} = \sum_{\substack{C \\ |C|=i \\ b(C)\geq i}} f_C f_{b(C)-i}$$

where C runs over rooted connected chord diagrams. Then

$$g_{k,i} = \sum_{\ell=1}^{i-1} (2\ell - 1) g_{1,i-\ell} g_{k-1,\ell} \quad \text{for } 2 \leq k \leq i$$

This is exactly the renormalization group equation on chord diagrams.

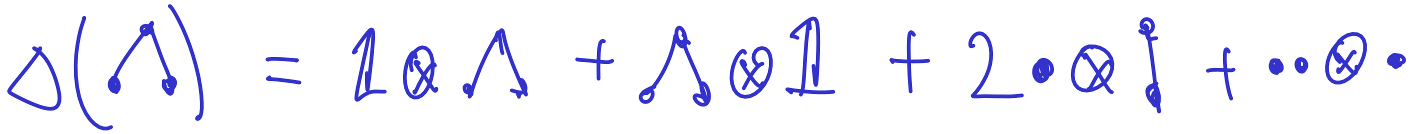
This gives one combinatorial view of the renormalization group equation.

Rooted trees revisited

Let $\mathcal{T}$ be rooted trees with no plane structure.
We had the polynomial algebra $\mathbb{Q}[\mathcal{T}]$. This can be turned into a Hopf algebra with the following coproduct

$$\Delta(t) = \sum_{\substack{C \subseteq V(t) \\ \text{antichain}}} \prod_{v \in C} t_v \otimes \left(t \setminus \prod_{v \in C} t_v \right)$$

where t_v is the subtree rooted at v.
Eg:



The counit is given by $\mathbb{I} \mapsto 1$ and $t \mapsto 0$ and the antipode is automatic by the grading. This is the **Connes-Kreimer** Hopf algebra $\mathcal{H}$.

Tree Feynman rules

Given $f, g : \mathcal{H} \to \mathbb{C}[L]$ define

$$f * g = m(f \otimes g)\Delta$$

Say f is Feynman rules if

$$f(L_1 + L_2) = f(L_1) * f(L_2)$$

subs. L_1+L_2 for L

Tree factorial

The simplest non-trivial tree Feynman rules come from the tree factorial

$$t! = \prod_{v \in V(t)} |t_v|$$

Eg:

The Feynman rules are

$$\phi(t) = \frac{L^{|t|}}{t!}$$

Green functions revisited

Given a tree class $\mathcal{T}$, form the Green function using tree factorial Feynman rules

$$G(x, L) = \phi\left(\sum_{t\in\mathcal{T}} t x^{|t|}\right) = \sum_{t\in\mathcal{T}} \frac{(xL)^{|t|}}{t!}$$

Eg:

This is strictly simpler than the physical case, but gives a universal combinatorial factor of the leading term. There is a similar story more generally for trees.

The renormalization group equation revisited

If $\mathcal{T}$ is physically reasonable (à la Foissy) then this $G(x, L)$ also satisfies the renormalization group equation.

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} + \gamma(x)\right) G(x, L) = 0$$

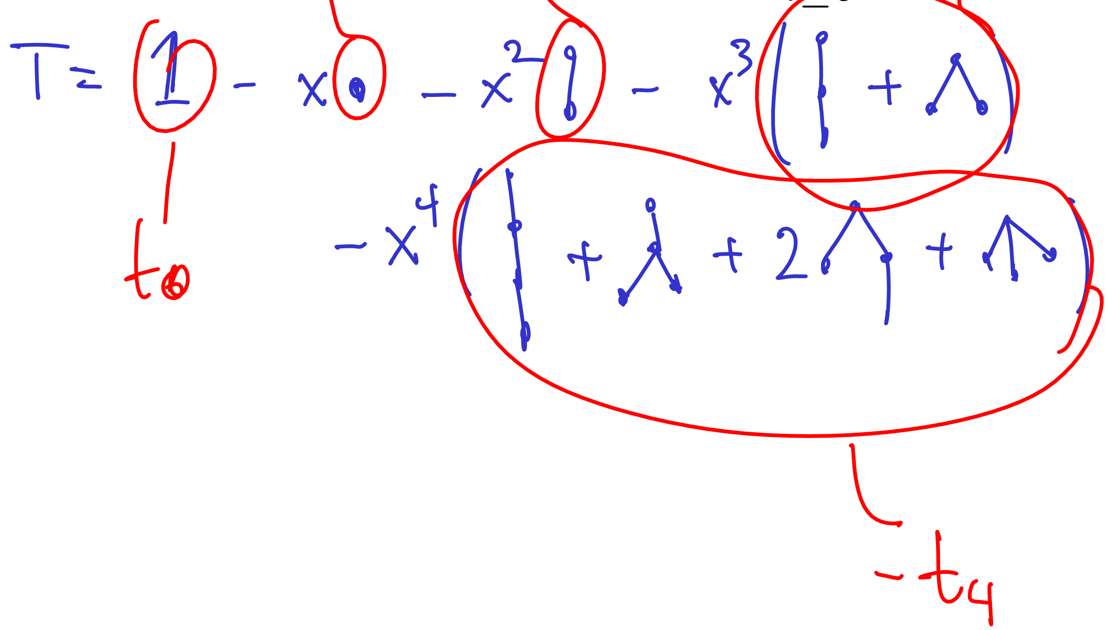
What does the renormalization group equation mean at the level of trees?

A series

Write

$$T(x) = \sum_{i \geq 0} t_i x^i$$

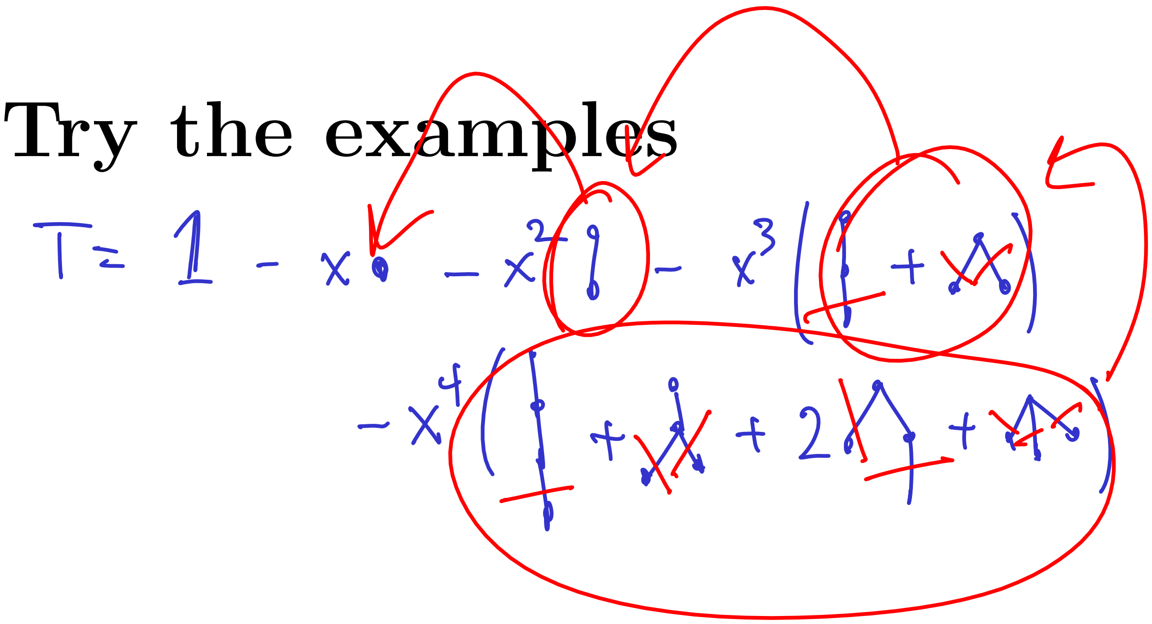
Eg:



Using subHopfness of $\mathcal{T}$ and the Feynman rule property of ϕ the renromalization group equation can be rephrased in terms of

$$c_{n,n-1} = \text{number of ways to get } t_{n-1} \text{ from } t_n \text{ by removing leaves}$$

Try the examples



1, 3, 5, 7 ...

In general

(joint with S. Bloch and D. Kreimer)
$G(x, L)$ satisfies the renormalization group equation if and only if $c_{n,n-1}$ is an arithmetic progression.

For general tree Feynman rules the same basic picture holds but there is a matrix not just a series.

This gives another combinatorial view of the renormalization group equation.

Conclusion

The renormalization group equation can be viewed combinatorially. The resulting recurrences are sometimes classical.

What else?

1. Higher renormalization group equations

2. Analogues for other types of combinatorial objects

Bonus slide