Feynman integrals and combinatorics

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Applied combinatorics graduate summer school
University of Saskatchewan, Saskatoon

Fourth and final lecture
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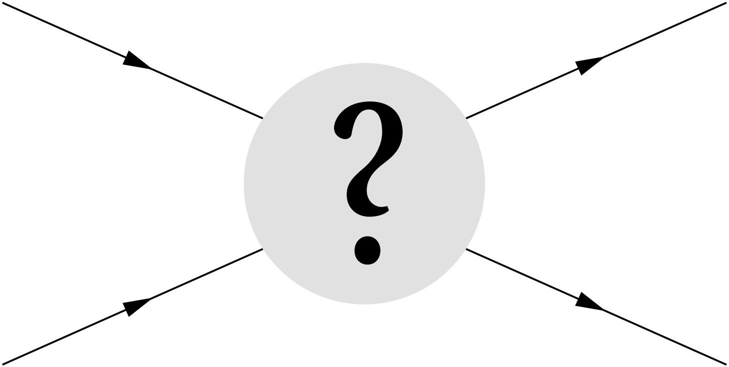
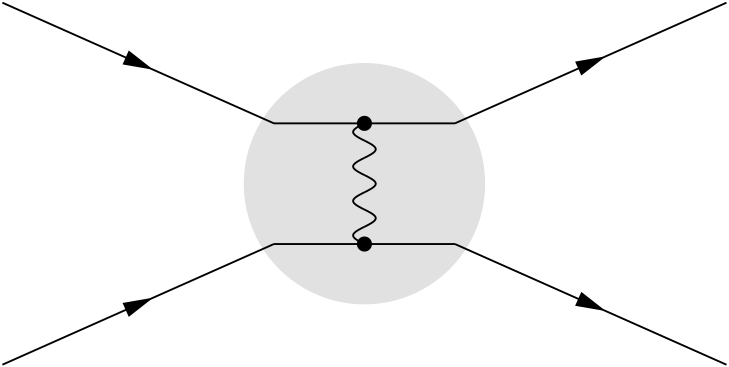
Perturbative Quantum Field Theory



Perturbative Quantum Field Theory



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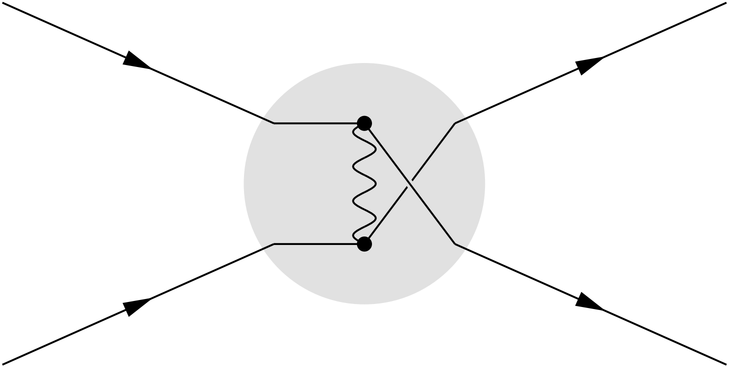

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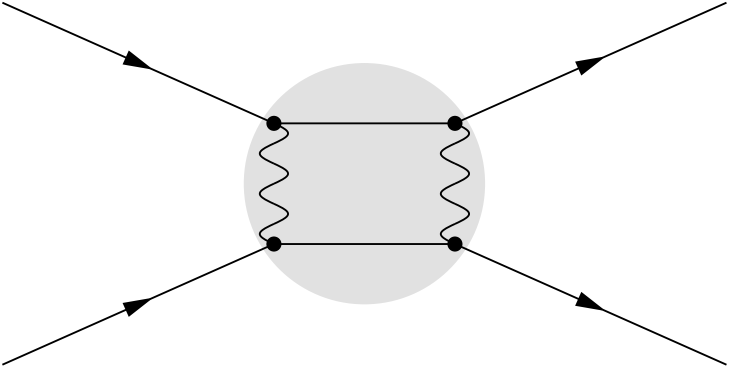

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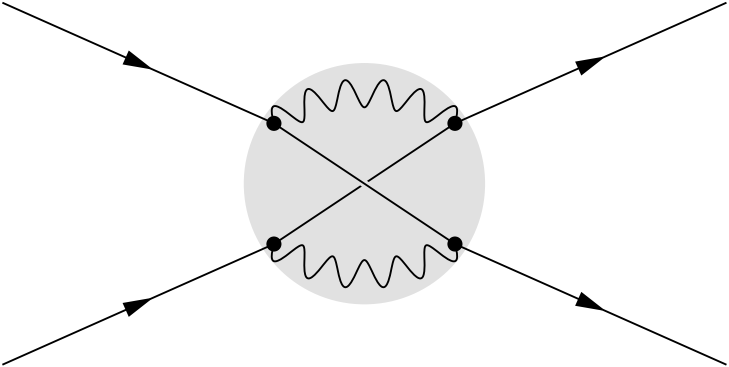

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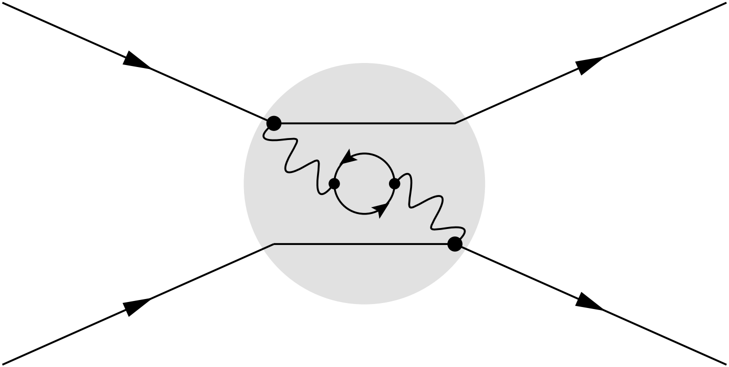

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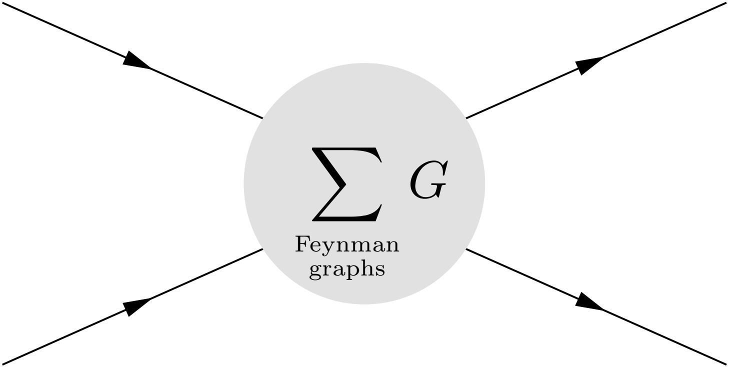

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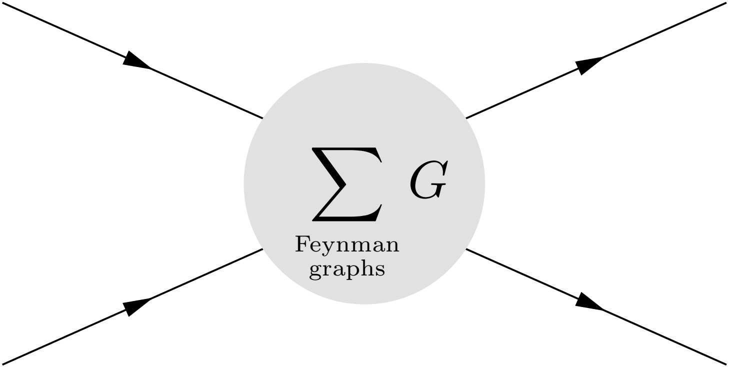

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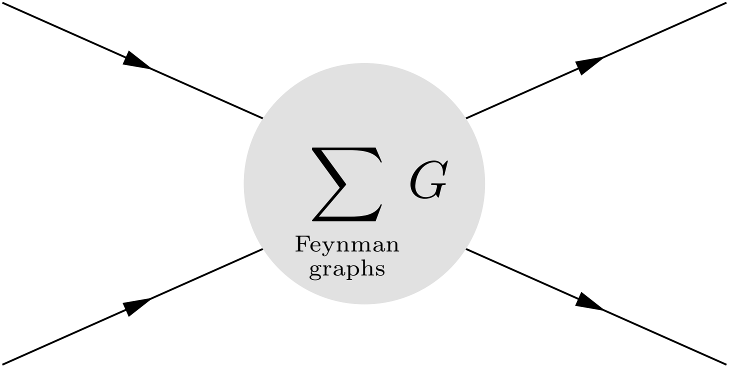

- each Feynman graph represents a Feynman integral $\Phi(G)$
- truncated sum $\sum_G \Phi(G)$ approximates the process
- very accurate measurements demand precise theoretical predictions
  Challenges: **number of graphs** & **complexity of integrals**

Feynman integrals: special functions and numbers

- Many (a few) FI are expressible via multiple polylogarithms

$$\mathrm{Li}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) = \sum_{0<k_1<\cdots<k_d} \frac{z_1^{k_1}\cdots z_d^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}}$$

- Frequent occurence of periods like multiple zeta values

$$\zeta_{n_1,\ldots,n_d} = \mathrm{Li}_{n_1,\ldots,n_d}(1,\ldots,1)$$

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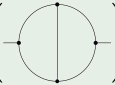
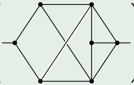
Example (massless propagators)

$\Phi\Big($  $\Big) = 6\zeta_3$, $\quad \Phi\Big($  $\Big) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{25056}{875}\zeta_2^4$

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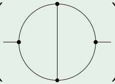
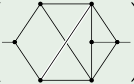
$\Phi\Big($  $\Big) = 6\zeta_3,\quad \Phi\Big($  $\Big) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{25056}{875}\zeta_2^4$

Questions

Why? When? How?

Schwinger parameters

With the *superficial degree of divergence* $\mathsf{sdd} = |E(G)| - D/2 \cdot \mathsf{loops}(G)$,

$$\Phi(G) = \Gamma(\mathsf{sdd}) \int_{(0,\infty)^E} \frac{\Omega}{\psi^{D/2}} \left(\frac{\psi}{\varphi}\right)^{\mathsf{sdd}}, \qquad \Omega = \delta(1-\alpha_N) \prod_{e \in E} \mathrm{d}\alpha_e$$

Graph polynomials:

$$\psi = \sum_T \prod_{e \notin T} \alpha_e \qquad \varphi = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e + \psi \sum_e m_e^2 \alpha_e$$

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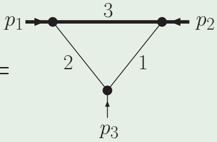
$$\psi = \alpha_1 + \alpha_2 + \alpha_3 \qquad \varphi = p_1^2 \alpha_2\alpha_3 + p_2^2 \alpha_1\alpha_3 + m^2 \alpha_3(\alpha_1 + \alpha_2 + \alpha_3)$$

$G =$ 

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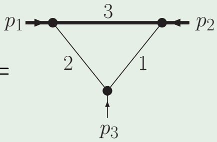
$G =$ 

$$\Phi(G) = \Gamma(1+\varepsilon) \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int_{(0,\infty)^E} \frac{\Omega}{\psi\varphi} \log^n \frac{\psi^2}{\varphi}$$

The period

In the logarithmic case ($\mathrm{sdd} = 0$), φ drops out.

Definition

If G is primitive and $\mathrm{sdd}(G) = 0$, its period is the number

$$\mathcal{P}(G) := \int \frac{\Omega}{\psi^2} \in \mathbb{R}_+.$$

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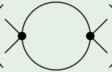
$\Phi\Bigg($  $\Bigg) = \iint_0^\infty \frac{\delta(1-\alpha_2)}{(\alpha_1+\alpha_2)^2}\, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2$

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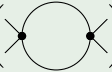
$\Phi\Big($  $\Big) = \iint_0^\infty \frac{\delta(1-\alpha_2)}{(\alpha_1+\alpha_2)^2}\, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 = 1$

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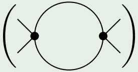
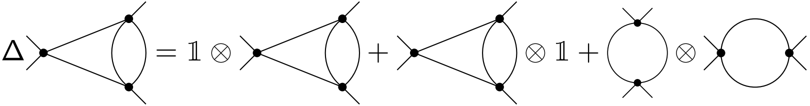


In $D = 4$, a graph is primitive $\Leftrightarrow$ it has no biconnected subgraphs with 2 or 4 external legs. Dunce's cap is not primitive:



The wheel with 3 spokes: K_4

$$\psi = \alpha_5\alpha_3\alpha_6 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_3\alpha_4 + \alpha_2\alpha_6\alpha_5 + \alpha_2\alpha_6\alpha_4 + \alpha_5\alpha_2\alpha_4$$
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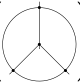
Contraction-deletion-formula (for e not a loop or bridge):

$$\psi_G = \alpha_e \psi_{G\setminus e} + \psi_{G/e}$$

First integrations:

$\mathcal{P}\Bigg($  $\Bigg) = \int \frac{\Omega}{\psi_G^2} = \int \frac{\Omega}{\psi_{G\setminus e}\psi_{G/e}} = \int \frac{\Omega}{W} \log \frac{\psi_{G\setminus e/f}\psi_{G\setminus f/e}}{\psi_{G\setminus ef}\psi_{G/ef}}$

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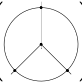
$\mathcal{P}\Bigg($  $\Bigg)$ $$= \int \frac{\Omega}{\psi_G^2} = \int \frac{\Omega}{\psi_{G\setminus e}\psi_{G/e}} = \int \frac{\Omega}{W} \log \frac{\psi_{G\setminus e/f}\psi_{G\setminus f/e}}{\psi_{G\setminus ef}\psi_{G/ef}}$$

where $W = (\alpha_6\alpha_4 + \alpha_4\alpha_5 + \alpha_3\alpha_5 + \alpha_5\alpha_6)^2$ factorizes!

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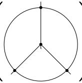
$\mathcal{P}$ () $= \int \frac{\Omega}{\psi_G^2} = \int \frac{\Omega}{\psi_{G\setminus e}\psi_{G/e}} = \int \frac{\Omega}{W} \log \frac{\psi_{G\setminus e/f}\psi_{G\setminus f/e}}{\psi_{G\setminus ef}\psi_{G/ef}}$

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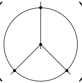
$\mathcal{P}$ () $= \cdots = 3 \int \frac{\Omega}{z(xy + xz + yz)} \log \frac{(x+z)(y+z)}{xy + xz + yz}$

Integration with hyperlogarithms

proposed by Brown, applications by Chavez & Duhr, Wißbrock, Anastasiou et. al.

Idea: integrate out one variable after the other, $f_n = \int_0^\infty f_{n-1} \, \mathrm{d}\alpha_n$.

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Strategy:

1. Write f_{n-1} in terms of hyperlogarithms:

$$f_{n-1} = \sum_{\vec{\sigma},\tau,k} \frac{G(\vec{\sigma}; \alpha_n)}{(\alpha_n - \tau)^k} \lambda_{\sigma,\tau,k} \quad \text{with } \vec{\sigma} \text{ and } \tau \text{ independent of } \alpha_n.$$

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3. Evaluate the limits

$$f_n := \int_0^\infty f_{n-1} \, \mathrm{d}\alpha_n = \lim_{\alpha_n \to \infty} F(\alpha_n) - \lim_{\alpha_n \to 0} F(\alpha_n).$$

Linear reducibility

We need that all partial integrals

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- condition on the polynomials ψ and φ only; independent of ε-order and expansion point $(D, \vec{a})_{\varepsilon=0} \in 2\mathbb{N} \times \mathbb{Z}^N$
- sufficient criteria: polynomial reduction algorithms (Brown)

Polynomial reduction

Denote alphabets (divisors) by sets S of irreducible polynomials.

Definition

Let S denote a set of polynomials $f = f^e\alpha_e + f_e$ linear in α_e. Then with $[f,g]_e := f^e g_e - f_e g^e$, S_e shall be the set of irreducible factors of

$$\{f^e, f_e \colon\ f \in S\} \quad \text{and} \quad \{[f,g]_e \colon\ f,g \in S\}\,.$$

Example (massless triangle)

$$S = \{\psi,\varphi\} = \{\alpha_1+\alpha_2+\alpha_3, \alpha_2\alpha_3 + z\bar{z}\alpha_1\alpha_3 + (1-z)(1-\bar{z})\alpha_1\alpha_2\}$$
$$[\varphi,\psi]_3 = (\alpha_1+\alpha_2)(\alpha_2+\alpha_1 z\bar{z}) - (1-z)(1-\bar{z})\alpha_1\alpha_2$$

Polynomial reduction

Denote alphabets (divisors) by sets S of irreducible polynomials.

Definition

Let S denote a set of polynomials $f = f^e\alpha_e + f_e$ linear in α_e. Then with $[f,g]_e := f^e g_e - f_e g^e$, S_e shall be the set of irreducible factors of

$$\{f^e, f_e\colon\; f \in S\} \quad \text{and} \quad \{[f,g]_e\colon\; f,g \in S\}\,.$$

Example (massless triangle)

$$S = \{\psi,\varphi\} = \{\alpha_1+\alpha_2+\alpha_3, \alpha_2\alpha_3 + z\bar{z}\alpha_1\alpha_3 + (1-z)(1-\bar{z})\alpha_1\alpha_2\}$$

$$[\varphi,\psi]_3 = (z\alpha_1+\alpha_2)(\bar{z}\alpha_1+\alpha_2)$$

$$S_3 = \{\alpha_1+\alpha_2, z\alpha_1+\alpha_2, \bar{z}\alpha_1+\alpha_2, z\bar{z}\alpha_1+\alpha_2, \alpha_1, \alpha_2, 1-z, 1-\bar{z}\}$$

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Lemma

If the singularities of F are cointained in S, then the singularities of $\int_0^\infty F \mathrm{d}\alpha_e$ are contained in S_e.

Polynomial reduction

Corollary (linear reducibility)

If all $S^k := (S^{k-1})_k$ are linear in α_{k+1}, then any MPL F with alphabet in S^0 integrates to a MPL $\int_0^\infty F \prod_{e=1}^n \mathrm{d}\alpha_e$ with alphabet in S^n.

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$$S_{3,2} = \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, z\bar{z}-1\}$$

This gives only very coarse upper bounds, for example $z\bar{z}-1$ is spurious: It drops out in $S_{2,3} \cap S_{3,2} = \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}\}$ because

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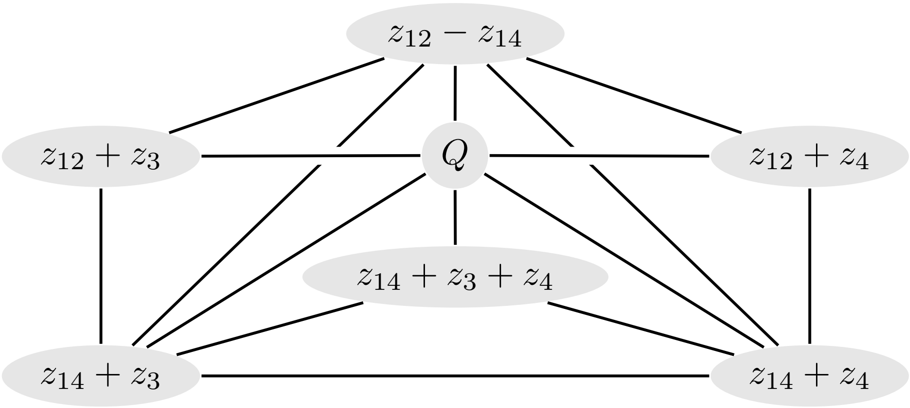
There are more sophisticated (and much more powerful) polynomial reduction alorithms (*Fubini* and several variants of *compatibility graphs*).

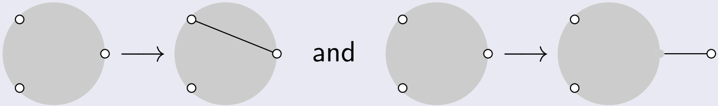
Compatibility graph of box-ladders



Linear reducibility: Known results

Definition (vertex-width 3 (Brown), 3-constructible (Schnetz))

The class of 3-point graphs including star, triangle and closed under



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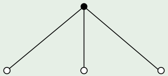

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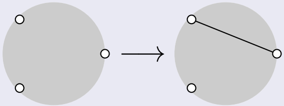

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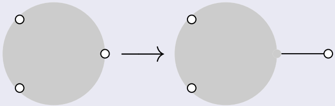

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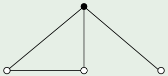

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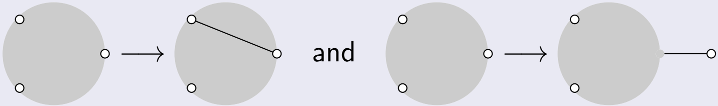

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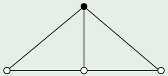

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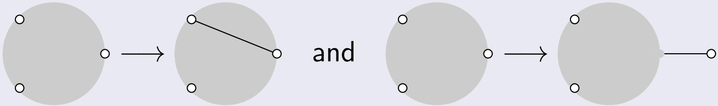

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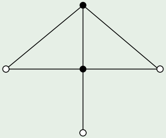

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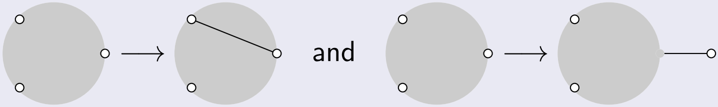

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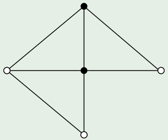

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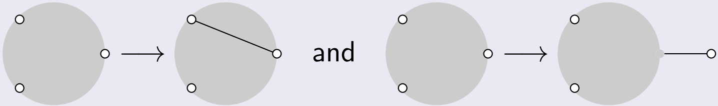

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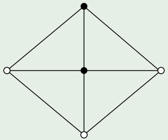

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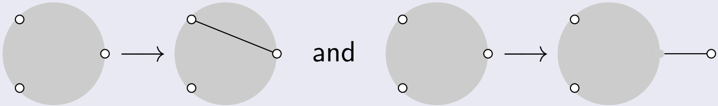

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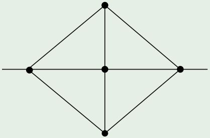

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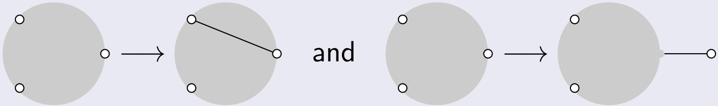

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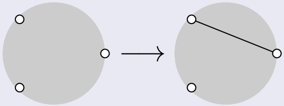

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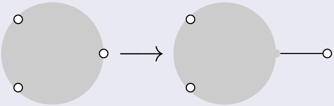
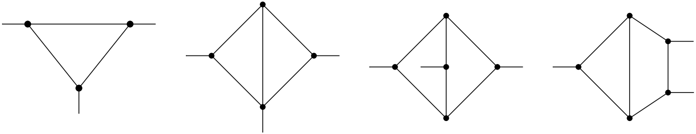
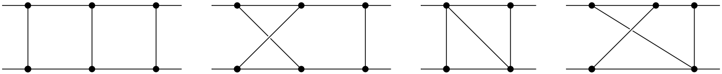


Linearly reducible:

1. 3-constructible massless propagators (Brown)
2. massless off-shell 3-point up to 2 loops (Chavez & Duhr):



3. massless on-shell 4-point up to 2 loops (Lüders):

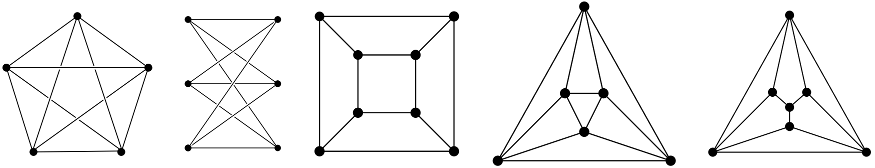


. . .

Forbidden minors for vertex-width 3

Theorem (Crump, Yeats et. al.)

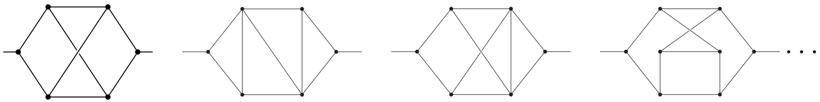
A simple, 3-connected graph G has vertex-width $\mathrm{vw}(G) = 3$ if and only if it contains none of $\{K_{3,3}, K_5, C, O, H\}$ as a minor.



K_5 $K_{3,3}$ C O H

New results

- all massless propagators up to 4 loops are linearly reducible



Theorem (generalizes Bierenbaum & Weinzierl from 2 to 4 loops)

All ε-expansion coefficients of $\leq$ 4-loop massless propagators $\mathbb{Q}$-linear combinations of MZV or alternating sums, for any $a_e \in \mathbb{Z} + \varepsilon\mathbb{Z}$. ***Effective!***

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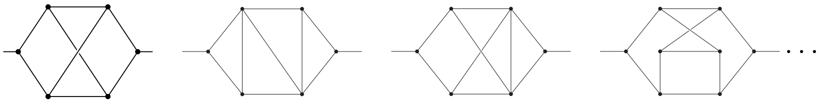

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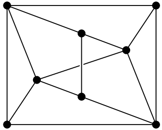
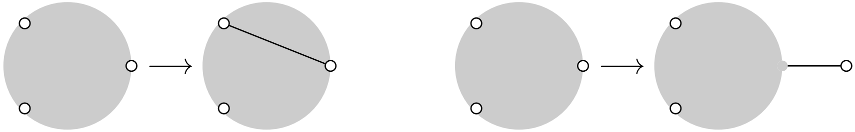
- all 7-loop primitive ϕ^4-periods (Broadhurst & Kreimer 1995) now known exactly (with Schnetz)

$P_{7,11} =$  $\Rightarrow$ MPL at $e^{i\pi/3}$ **(not MZV!)**

Linearly reducible only after change of variables

Linear reducibility: Infinite families

- 3-constructible graphs (as 3-point functions)



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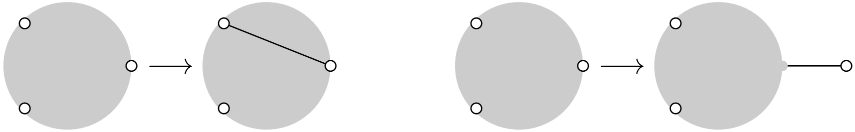
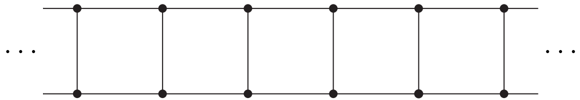


- minors of ladder-boxes (up to 2 legs off-shell)



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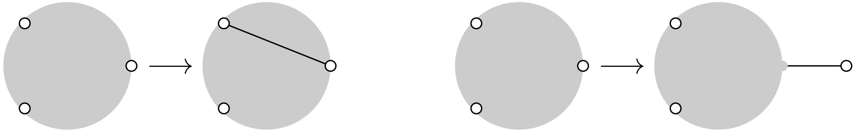

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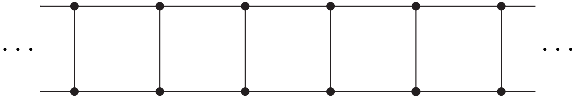
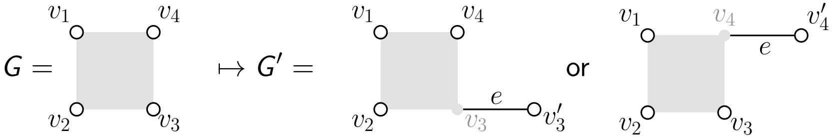


- Techniques:
  1. forest functions (inverse Laplace transform of Φ)
  2. recursive integral equations
  3. improved polynomial reduction

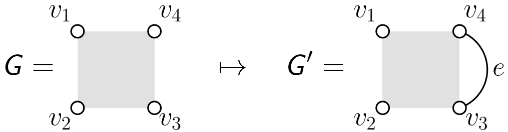
4-point recursions

Start with the box and repeat, in any order:

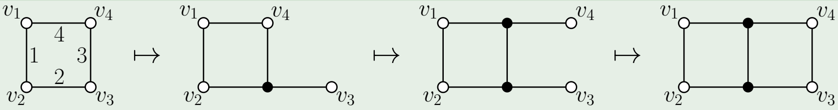
- Appending a vertex:



- Adding an edge:



Example



Forest polynomials

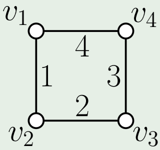
Definition

Spanning forest polynomial $\Phi^{A,B} := \sum_F \prod_{e \notin F} \alpha_e$ over 2-forests F which separate the vertices A and B.

$$f_{12} := \Phi^{\{1,2\},\{3,4\}} \qquad f_3 := \Phi^{\{3\},\{1,2,4\}}$$

$$f_{14} := \Phi^{\{1,4\},\{2,3\}} \qquad f_4 := \Phi^{\{4\},\{1,2,3\}}$$

Example



$$\psi = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \qquad f_{12} = \alpha_2\alpha_4 \qquad f_3 = \alpha_2\alpha_3$$

$$f_{14} = \alpha_1\alpha_3 \qquad f_4 = \alpha_3\alpha_4$$

$$\varphi = \mathcal{F} = (p_1 + p_2)^2 f_{12} + (p_1 + p_4)^2 f_{14} + p_3^2 f_3 + p_4^2 f_4$$

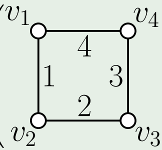
Restricting forest polynomials

Definition

$$F_G(z) := \int_{\mathbb{R}_+^E} \psi_G^{-D/2} \cdot \delta^{(4)}\left(\frac{f}{\psi} - z\right) \prod_{e \in E} \alpha_e^{a_e - 1}\, \mathrm{d}\alpha_e \qquad (\mathbb{R}_+^4 \longrightarrow \mathbb{R}_+)$$

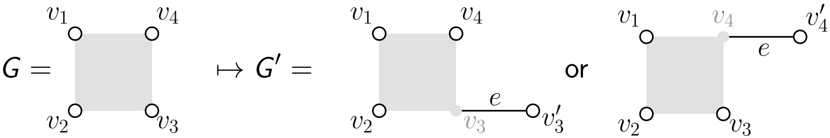
Example $(a_1 = a_2 = a_3 = a_4 = 1)$

$F\Big($



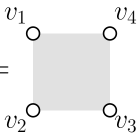
$; z\Big) = \begin{cases} \dfrac{1}{z_3 z_4} & (D = 4) \\[2ex] \dfrac{z_{12}}{\big[\underbrace{z_{12}(z_{14} + z_3 + z_4) + z_3 z_4}_{Q}\big]^2} & (D = 6) \end{cases}$

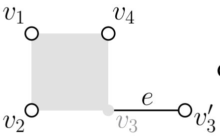
Appending a vertex

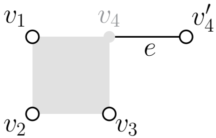


Using $(f'_{12}, f'_{14}, f'_3, f'_4, \psi') = (f_{12}, f_{14}, f_3, f_4 + \alpha_e \psi, \psi)$ where $x = \alpha_e$,

$$F_{G'}(z) = \int_0^{z_4} F_G(z_{12}, z_{14}, z_3, z_4 - x) \cdot x^{a_e - 1} \, \mathrm{d}x$$

$G =$ 

$\mapsto G' =$ 

or 

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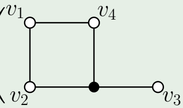
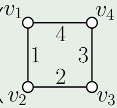
Example ($D = 6$ and $a_e = 1$)

$F\Bigg($ 

$; z\Bigg) = \int_0^{z_3} F\Bigg($ 

$; z_{12}, z_{14}, z'_3, z_4\Bigg) \, \mathrm{d}z'_3$

Appending a vertex

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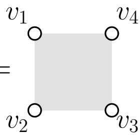

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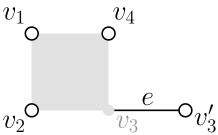

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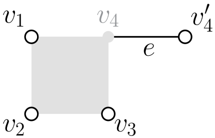

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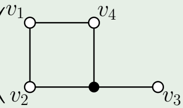
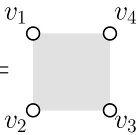
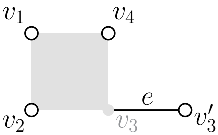
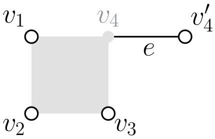


$; z\Big) = \int_0^{z_3} \frac{z_{12} \, \mathrm{d}z'_3}{[z_{12}(z_{14} + z_3 + z_4) + z_3 z_4]^2} = \frac{z_3}{(z_{14} + z_4) \cdot Q}$

Appending a vertex

$G =$ 

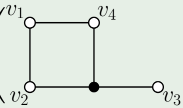
$\mapsto G' =$ 

or 

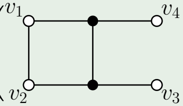
Using $(f'_{12}, f'_{14}, f'_3, f'_4, \psi') = (f_{12}, f_{14}, f_3, f_4 + \alpha_e \psi, \psi)$ where $x = \alpha_e$,

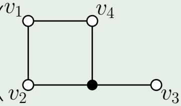
$$F_{G'}(z) = \int_0^{z_4} F_G(z_{12}, z_{14}, z_3, z_4 - x) \cdot x^{a_e - 1} \, \mathrm{d}x$$

Example ($D = 6$ and $a_e = 1$)

$F\Bigg($ 

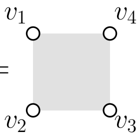
$; z\Bigg) = \int_0^{z_3} \frac{z_{12} \, \mathrm{d}z'_3}{[z_{12}(z_{14} + z_3 + z_4) + z_3 z_4]^2} = \frac{z_3}{(z_{14} + z_4) \cdot Q}$

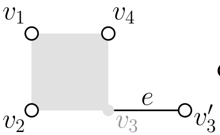
$F\Bigg($ 

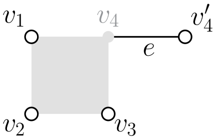
$; z\Bigg) = \int_0^{z_4} F\Bigg($ 

$; z_{12}, z_{14}, z_3, z'_4\Bigg) \, \mathrm{d}z'_4$

Appending a vertex

$G =$ 

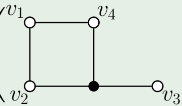
$\mapsto G' =$ 

or 

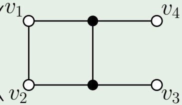
Using $(f'_{12}, f'_{14}, f'_3, f'_4, \psi') = (f_{12}, f_{14}, f_3, f_4 + \alpha_e \psi, \psi)$ where $x = \alpha_e$,

$$F_{G'}(z) = \int_0^{z_4} F_G(z_{12}, z_{14}, z_3, z_4 - x) \cdot x^{a_e - 1}\, \mathrm{d}x$$

Example ($D = 6$ and $a_e = 1$)

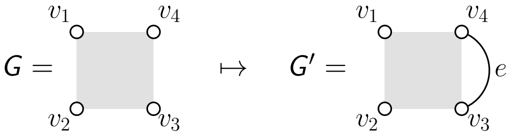
$F\Big($ 

$; z\Big) = \int_0^{z_3} \frac{z_{12}\, \mathrm{d}z_3'}{[z_{12}(z_{14} + z_3 + z_4) + z_3 z_4]^2} = \frac{z_3}{(z_{14} + z_4) \cdot Q}$

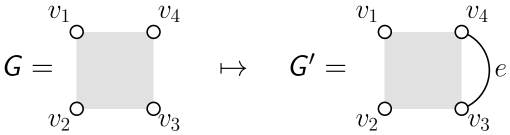
$F\Big($ 

$; z\Big) = \frac{1}{z_{12} - z_{14}} \ln \frac{z_{12}(z_3 + z_{14})(z_4 + z_{14})}{z_{14} \cdot Q}$

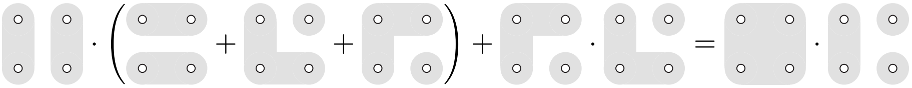
Adding an edge



Adding an edge

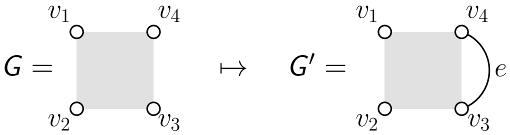


Dodgson-identities between spanning forest polynomials:

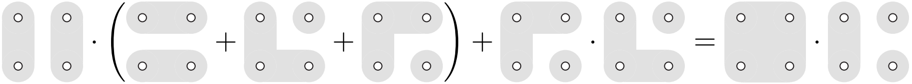


$$f_{12}\left(f_{14}+f_3+f_4\right)+f_3 f_4 = Q(f) = \psi \cdot \Phi^{\{1,2\},\{3\},\{4\}}$$

Adding an edge



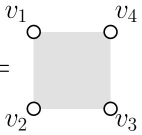
Dodgson-identities between spanning forest polynomials:

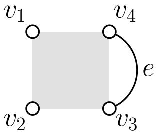


$$f_{12}\,(f_{14} + f_3 + f_4) + f_3 f_4 = Q(f) = \psi \cdot \Phi^{\{1,2\},\{3\},\{4\}}$$

$$F_{G'}(z) = Q^{a_e + \mathrm{sdd} - D} \int_0^{z_{12}} x^{D/2 - a_e - 1} \left[Q^{D/2 - \mathrm{sdd}} \cdot F_G \right]_{z_{12} = z_{12} - x} \mathrm{d}x$$

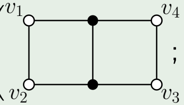
Adding an edge

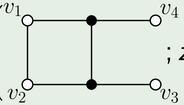
$G =$ 

$\mapsto$ $G' =$ 

$$F_{G'}(z) = Q^{a_e + \mathrm{sdd} - D} \int_0^{z_{12}} x^{D/2 - a_e - 1} \left[Q^{D/2 - \mathrm{sdd}} \cdot F_G \right]_{z_{12} = z_{12} - x} \mathrm{d}x$$

Example ($D = 6$ and $a_e = 1$)

$F\Bigg($ 

$; z\Bigg) = \frac{1}{Q^2} \int_0^{z_{12}} F\Bigg($ 

$; z_{12} - x, z_{14}, z_3, z_4\Bigg)\, x\,\mathrm{d}x$

$$= \frac{z_{12} - z_{14}}{Q^2} \left[\ln \frac{Q}{z_3 z_4} \ln \frac{(z_{14} + z_3)(z_{14} + z_4)}{z_{14}(z_{14} + z_3 + z_4)} - \mathrm{Li}_2 \left(\frac{z_3 z_4 (z_{14} - z_{12})}{z_{14} Q} \right) \right]$$

$$+ \frac{z_{12} - z_{14}}{Q^2} \mathrm{Li}_2 \left(\frac{z_3 z_4}{Q} \right) + \frac{z_{12}}{Q^2} \ln \frac{z_{14} z_3 z_4}{z_{12}(z_{14} + z_3)(z_{14} + z_4)} - \frac{\ln(z_3 z_4 / Q)}{Q(z_{14} + z_3 + z_4)}$$

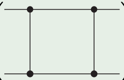
Kinematics from forest functions

$$\varphi = \mathcal{F} = (p_1 + p_2)^2 f_{12} + (p_1 + p_4)^2 f_{14} + p_3^2 f_3 + p_4^2 f_4$$

Corollary

$$\Phi(G) = \frac{\Gamma(\mathrm{sdd})}{\prod_e \Gamma(a_e)} \int_0^\infty \frac{F_G(z)\,\Omega}{\left[(p_1+p_2)^2 z_{12} + (p_1+p_4)^2 z_{14} + p_3^2 z_3 + p_4^2 z_4\right]^{\mathrm{sdd}}}$$

Example (kinematics: $s = (p_1+p_2)^2$ and $u = (p_1+p_4)^2$)

Φ ()

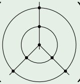
$$= \int_0^\infty \frac{\mathrm{d}z_{12}}{s z_{12} + u} \int_0^\infty \mathrm{d}z_3 \int_0^\infty \mathrm{d}z_4 \, \frac{z_{12}}{[z_{12}(1+z_3+z_4) + z_4 z_3]^2}$$

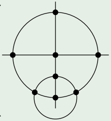
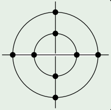
$$= \int_0^\infty \frac{\mathrm{d}z_{12}\ \ln z_{12}}{(s z_{12} + u)(z_{12} - 1)} = \frac{\pi^2 + \ln^2(s/u)}{2(s+u)}$$

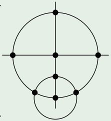
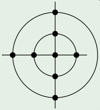
Periods of cocommutative graphs

The Feynman period of log. div. graphs depends on the renormalization scheme/point. Cocommutative graphs are an exception.

Example (wheels in wheels)

$\mathcal{P}\Big($  $\Big) = 72\zeta_3^2 - \frac{189}{2}\zeta_7$

$\mathcal{P}\Big($  $-$  $\Big) = 72\zeta_3^3$

$\mathcal{P}\Big($  $+$  $\Big) = 480\zeta_3\zeta_5 - 40\zeta_3^3 - \frac{4730}{9}\zeta_9$

HyperInt

- open source Maple program
- integration of hyperlogarithms
- transformations of arguments (functional equations)
- polynomial reduction
- graph polynomials
- symbolic computation of constants (no numerics)

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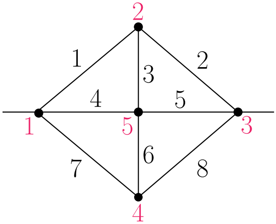
Example

```
> read "HyperInt.mpl":
> hyperInt(polylog(2,-x)*polylog(3,-1/x)/x,x=0..infinity):
> fibrationBasis(%);
```

$$\frac{8}{7}\zeta_2^3$$

computes $\int_0^\infty \mathrm{Li}_2(-x)\,\mathrm{Li}_3(-1/x)\mathrm{d}x = \frac{8}{7}\zeta_2^3$.

HyperInt: propagator




```
> E := [[2,1],[2,3],[2,5],[5,1],[5,3],[5,4],[4,1],[4,3]]:
> psi := graphPolynomial(E):
> phi := secondPolynomial(E, [[1,1], [3,1]]):
> add((epsilon*log(psi^5/phi^4))^n/n!,n=0..2)/psi^2:
> hyperInt(eval(%,x[8]=1), [seq(x[n],n=1..7)]):
> collect(fibrationBasis(%), epsilon);
```

$$\left(254\zeta_7 + 780\zeta_5 - 200\zeta_2\zeta_5 - 196\zeta_3^2 + 80\zeta_2^3 - \frac{168}{5}\zeta_2^2\zeta_3\right)\varepsilon^2 + \left(-28\zeta_3^2 + 140\zeta_5 + \frac{80}{7}\zeta_2^3\right)\varepsilon + 20\zeta_5$$

HyperInt: triangle

Graph polynomials:

```
> E:=[[1,2],[2,3],[3,1]]:
> M:=[[3,1],[1,z*zz],[2,(1-z)*(1-zz)]]:
> psi:=graphPolynomial(E):
> phi:=secondPolynomial(E,M):
```

Integration:

```
> hyperInt(eval(1/psi/phi,x[3]=1),[x[1],x[2]]):
> factor(fibrationBasis(%,[z,zz]));
```

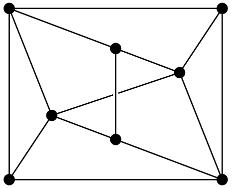
$$(\mathrm{Hlog}(1;z)\,\mathrm{Hlog}(0;zz) - \mathrm{Hlog}(0;z)\,\mathrm{Hlog}(1;zz) + \mathrm{Hlog}(0,1;zz) - \mathrm{Hlog}(1,0;zz) + \mathrm{Hlog}(1,0;z) - \mathrm{Hlog}(0,1;z))/(z-zz)$$

Polynomial reduction:

```
> L[{}]:=[{psi,phi}, {{psi,phi}}]:
> cgReduction(L):
> L[{x[1],x[2]}][1];
```

$$\{-1+z, -1+zz, -zz+z\}$$

Massless ϕ^4 theory: primitive sixth roots of unity

$P_{7,11} =$  is not linearly reducible!

Tenth denominator:

$$\begin{aligned} d_{10} = {} & \alpha_2\alpha_4^2\alpha_1 + \alpha_2\alpha_4^2\alpha_3 - \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_2^2\alpha_4\alpha_1 + \alpha_2^2\alpha_4\alpha_3 \\ & - 2\alpha_2\alpha_3^2\alpha_4 - \alpha_2^2\alpha_3^2 - 2\alpha_2^2\alpha_3\alpha_1 - 2\alpha_2\alpha_3^2\alpha_1 - \alpha_3^2\alpha_4^2 \\ & - 2\alpha_3^2\alpha_4\alpha_1 - \alpha_2^2\alpha_1^2 - 2\alpha_2\alpha_3\alpha_1^2 - \alpha_3^2\alpha_1^2. \end{aligned}$$

Change variables: $\alpha_3 = \frac{\alpha_3'\alpha_1}{\alpha_1+\alpha_2+\alpha_4}$, $\alpha_4 = \alpha_4'(\alpha_2 + \alpha_3')$ and $\alpha_1 = \alpha_1'\alpha_4'$,

$$d_{10}' = (\alpha_2 + \alpha_3')(\alpha_2 + \alpha_2\alpha_4' - \alpha_1')(\alpha_1'\alpha_4' + \alpha_2 + \alpha_2\alpha_4' + \alpha_3'\alpha_4')$$

Final result: *not a multiple zeta value*, instead MPL at $e^{i\pi/3}$

$\sqrt{3}\,\mathcal{P}(P_{7,11})$

$$= \operatorname{Im}\left(\tfrac{19\,285}{6}\zeta_9 \operatorname{Li}_2 - \tfrac{1029}{2}\zeta_7 \operatorname{Li}_4 + 240\zeta_3^2(9\operatorname{Li}_{2,3} - 7\zeta_3 \operatorname{Li}_2)\right) - \tfrac{93\,824}{9675}\pi^3\zeta_{3,5}$$

$$+ \tfrac{2592}{215}\operatorname{Im}\Big(36\operatorname{Li}_{2,2,2,5} + 27\operatorname{Li}_{2,2,3,4} + 9\operatorname{Li}_{2,2,4,3} + 9\operatorname{Li}_{2,3,2,4} + 3\operatorname{Li}_{2,3,3,3}$$

$$- 43\zeta_3(\operatorname{Li}_{2,3,3} + 3\operatorname{Li}_{2,2,4})\Big) - \tfrac{96\,393\,596\,519\,864\,341\,538\,701\,979}{790\,371\,465\,315\,684\,594\,157\,620\,000}\pi^{11}$$

$$+ \tfrac{216}{14\,755\,731\,798\,995}\operatorname{Im}\Big(2\,539\,186\,130\,125\,890\operatorname{Li}_8\zeta_3 - 1\,269\,593\,065\,062\,945\operatorname{Li}_{2,9}$$

$$-413\,965\,317\,054\,502\operatorname{Li}_6\zeta_5 - 996\,412\,983\,391\,539\operatorname{Li}_{3,8}$$

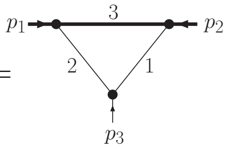
$$-546\,306\,741\,059\,841\operatorname{Li}_{4,7} - 156\,228\,639\,992\,955\operatorname{Li}_{5,6}\Big)$$

$$+ \tfrac{2592}{10\,945\,435}\pi^2\operatorname{Im}\Big(287\,205\operatorname{Li}_{2,7} - 574\,410\operatorname{Li}_6\zeta_3 + 55\,687\operatorname{Li}_{4,5} + 168\,941\operatorname{Li}_{3,6}\Big)$$

$$+ \pi\left(\tfrac{11\,613\,751}{9030}\zeta_5^2 + \tfrac{267\,067}{602}\zeta_{3,7} - \tfrac{31\,104}{215}\operatorname{Re}(3\operatorname{Li}_{4,6} + 10\operatorname{Li}_{3,7})\right)$$

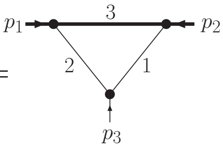
Abbreviation: $\operatorname{Li}_{n_1,\ldots,n_r} := \operatorname{Li}_{n_1,\ldots,n_r}(e^{i\pi/3})$

Divergences in Schwinger parameters

$G =$ 

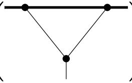
$$\psi = \alpha_1 + \alpha_2 + \alpha_3$$
$$\varphi = p_1^2\alpha_2\alpha_3 + p_2^2\alpha_1\alpha_3 + m^2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3)$$

Divergences in Schwinger parameters

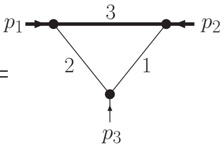
$G =$ 

$$\psi = \alpha_1 + \alpha_2 + \alpha_3$$
$$\varphi = p_1^2 \alpha_2 \alpha_3 + p_2^2 \alpha_1 \alpha_3 + m^2 \alpha_3 (\alpha_1 + \alpha_2 + \alpha_3) = \alpha_3 \tilde{\varphi}$$

In $D = 4 - 2\varepsilon$, subdivergence $\int_0 \frac{\mathrm{d}\alpha_3}{\alpha_3}$ at $\varepsilon = 0$:

Φ_D () $= \Gamma(1+\varepsilon) \int \frac{\Omega}{\psi^{1-2\varepsilon} \tilde{\varphi}^{1+\varepsilon} \alpha_3^{1+\varepsilon}}$

Divergences in Schwinger parameters

$G =$ 

$$\psi = \alpha_1 + \alpha_2 + \alpha_3$$
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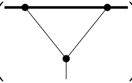
In $D = 4 - 2\varepsilon$, subdivergence $\int_0 \frac{\mathrm{d}\alpha_3}{\alpha_3}$ at $\varepsilon = 0$:

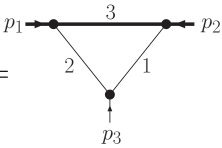
$$\Phi_D\left(\right) = \Gamma(1+\varepsilon)\int \frac{\Omega}{\psi^{1-2\varepsilon}\tilde{\varphi}^{1+\varepsilon}\alpha_3^{1+\varepsilon}}$$



Regularization: integrate by parts

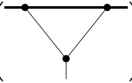
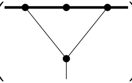
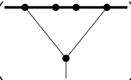
$$\int \frac{\Omega}{\psi^{1-2\varepsilon}\tilde{\varphi}^{1+\varepsilon}\alpha_3^{1+\varepsilon}} = \frac{-\alpha_3^{-\varepsilon}}{\varepsilon\psi^{1-2\varepsilon}\tilde{\varphi}^{1+\varepsilon}}\Bigg|_{\alpha_3=0}^{\infty} + \frac{1}{\varepsilon}\int \frac{\Omega}{\alpha_3^{\varepsilon}}\frac{\partial}{\partial\alpha_3}\psi^{-1+2\varepsilon}\tilde{\varphi}^{-1-\varepsilon}$$
$$= \frac{1}{\varepsilon}\int \frac{\Omega\alpha_3}{\psi^{1-2\varepsilon}\varphi^{1+\varepsilon}}\left[\frac{2\varepsilon-1}{\psi} - \frac{(1+\varepsilon)\alpha_3 m^2}{\varphi}\right]$$

Divergences in Schwinger parameters

$G =$ 

$$\psi = \alpha_1 + \alpha_2 + \alpha_3$$
$$\varphi = p_1^2\alpha_2\alpha_3 + p_2^2\alpha_1\alpha_3 + m^2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_3\tilde{\varphi}$$

In $D = 4 - 2\varepsilon$, subdivergence $\int_0 \frac{\mathrm{d}\alpha_3}{\alpha_3}$ at $\varepsilon = 0$:

$\Phi_D \Bigg($  $\Bigg) = \left(2 - \frac{1}{\varepsilon}\right) \Phi_{D+2} \Bigg($  $\Bigg) - \frac{2m^2}{\varepsilon} \Phi_{D+2} \Bigg($  $\Bigg)$

Regularization: integrate by parts

$$\int \frac{\Omega}{\psi^{1-2\varepsilon}\tilde{\varphi}^{1+\varepsilon}\alpha_3^{1+\varepsilon}} = \frac{-\alpha_3^{-\varepsilon}}{\varepsilon\psi^{1-2\varepsilon}\tilde{\varphi}^{1+\varepsilon}}\Bigg|_{\alpha_3=0}^{\infty} + \frac{1}{\varepsilon}\int \frac{\Omega}{\alpha_3^{\varepsilon}} \frac{\partial}{\partial\alpha_3} \psi^{-1+2\varepsilon}\tilde{\varphi}^{-1-\varepsilon}$$
$$= \frac{1}{\varepsilon}\int \frac{\Omega\alpha_3}{\psi^{1-2\varepsilon}\varphi^{1+\varepsilon}} \left[\frac{2\varepsilon - 1}{\psi} - \frac{(1+\varepsilon)\alpha_3 m^2}{\varphi}\right]$$

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Theorem

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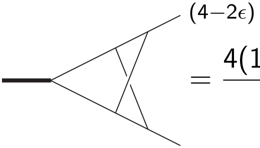
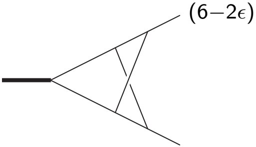
Corollary (IBP, Euclidean kinematics)

One can choose master integrals to be scalar and free of subdivergences, given that one allows for shifted dimensions $D+2$, $D+4$, $\ldots$ *and dots.*

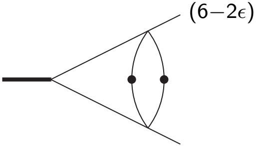
Two-loop non-planar form factor: expansion in primitives



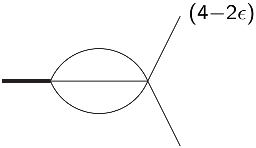
$$= \frac{4(1-\epsilon)(3-4\epsilon)(1-4\epsilon)}{\epsilon s^2}$$



$$- \frac{10 - 65\epsilon + 131\epsilon^2 - 74\epsilon^3}{\epsilon^3 s^2}$$



$$- \frac{14 - 119\epsilon + 355\epsilon^2 - 420\epsilon^3 + 172\epsilon^4}{(1-2\epsilon)\epsilon^3 s^3}$$



Double box: IBP reduction to primitive master integrals