THE RIEMANN MAPPING PROBLEM IN $\mathbb{C}^{n\geq 2}$
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Abstract. In this note we study biholomorphic maps of domains in $\mathbb{C}^n$ and prove the biholomorphic inequivalence of unit ball and unit polydisc if $n \geq 2$.

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a mapping defined by $F(z_1, \ldots, z_n) = (F_1(z_1, \ldots, z_n), \ldots, F_n(z_1, \ldots, z_n))$ where $F_i : \mathbb{C}^n \to \mathbb{C}$ are co-ordinate functions. We say F is holomorphic if F_i is holomorphic for each i. F is said to be biholomorphic if F^{-1} exists and F^{-1} holomorphic. Similarly we say U,$V \subset \mathbb{C}^n$ are biholomorphic if $\exists F : U \to V$ which is biholomorhic. We define the unit ball $\mathbb{B}(n)$ and the unit polydisc $\mathbb{D}(n)$ in the following manner :

$$\mathbb{B}(n) = \{z \in \mathbb{C}^n \mid \|z\|_2 < 1\} \text{ and } \mathbb{D}(n) = \{z \in \mathbb{C}^n \mid \|z_i\| < 1, i \in \{1, \ldots, n\}\}$$

An important section in every introductory course on function theory in one variable is Riemann's Mapping Theorem, which states that every simply connected domain D properly contained in $\mathbb{C}$ is biholomorphically equivalent to the open unit disc. This is especially remarkable, because a purely topological property simple connectedness implies a very restrictive analytical property. In more than one variable, however, there are simply connected domains which are not biholomorphically equivalent. In particular, this holds for unit ball and unit polydisc. This fact was first discovered by H. Poincare in 1907 [2] by proving that the groups of holomorphic automorphisms of $\mathbb{B}(n)$ and $\mathbb{D}(n)$ are not isomorphic if $n \geq 2$.

Proposition 0.1. *$\mathbb{B}(n)$ and $\mathbb{D}(n)$ are not biholomorphic for $n \geq 2$*

Proof. Let us first define the sets $Aut(D) = \{\Phi : D \to D \mid \Phi \text{ biholomorphic}\}$ and for $p \in D$, $Aut(D; p) = \{\Phi \in Aut(D) \mid \Phi(p) = p\}$.

We need the following lemmas (whose proof is left as an exercise) in order to prove the main result.

Lemma 1. *If D, E are biholomorphic, then $Aut(D) \cong Aut(E)$.*

Lemma 2. *If D, E are biholomorphic, $p \in D \cap E \neq \phi$, $Aut(E)$ acts transitively on E then $Aut(D; p) \cong Aut(E; p)$.*

Note that $Aut(\mathbb{B}(n))$ has the property that it acts transitively on $\mathbb{B}(n)$ and this can be viewed by defining an element in the automorphism group denoted by $\Phi_{x_0}(z)$ where

$$\Phi_{x_0}(z) = \frac{x_0 - \pi_{x_0}(z) - \sqrt{1 - \|x_0\|^2}\pi^{\perp}(z)}{1 - < x_0, z >}$$

Along with the above property we also have $0 \in \mathbb{B}(n) \cap \mathbb{D}(n)$. So in order to prove our proposition we need to show that $Aut(\mathbb{B}(n); 0) \ncong Aut(\mathbb{D}(n); 0)$.
Indeed one can achieve this by doing some computation with these groups. We have two important theorems as follows :-

Theorem 0.2. $Aut(\mathbb{B}(n); 0) \cong U_n((C))$

Proof. See Chap 3 in [4]. $\square$

Theorem 0.3. $Aut(\mathbb{D}(n); 0) \cong S_1 \times \cdots \times S_1 \bowtie Symm(n)$

Proof. See [3]. $\square$

Now we can clearly see that $Aut(\mathbb{B}(n); 0) \ncong Aut(\mathbb{D}(n); 0)$ for $n \geq 2$ as $Aut(\mathbb{D}(n); 0)$ has n^2 parameters. On the other hand $Aut(\mathbb{B}(n); 0) \cong U_n((C))$ which implies it has number of parameters $= 2n^2 - \frac{n(n+1)}{2} = \frac{3n^2-n}{2} \neq n^2$ for $n \geq 2$. This completes the proof of the proposition. $\square$

One can also see an outline of the proof of the above proposition in [1].

References

[1] Ohsawa T. *Analysis of Several Complex Variables*
[2] Poincare, H, *Les fonctions analytiques de deux variables et la r?epresentation conforme*, Rend. Circ. Mat. Palermo 23 (1907), 185-220
[3] Rudin W. *Function Theory in Polydiscs*
[4] Scheidemann V.*Introduction to Complex Analysis in Several Variables* Basel; Boston: Birkhuser Verlag, 2005.
[5] Taylor Joseph L., *Several Complex Variables with connections to Algebraic Geometry and Lie Groups* Volume 46 AMS, Providence, Rhode Island.