Summary: Weierstrass theorems

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The Weierstrass theorems are a set of powerful theorems we can use to provide inductive arguments on germs, which shall be defined next, and unlock inverse and implicit function theorems.

Definition 0.1. Let X be a topological space, $x \in X$ and U, V neighbourhoods of x.

If f is a function defined on U, g is a function defined on V and $f(y) = g(y)$ for all $y \in W \subset U \cap V$ then f is *equivalent* to g ($f \equiv g$) at x.

We call $\underline{f} = \{g : f \equiv g \text{ at } x\}$ the *germ* of f at x.

We note that the germs of complex functions at x is an algebra over $\mathbb{C}$. We shall denote this algebra by ${}_nH_\lambda$.

Now we shall introduce a multitude of propositions, to prepare us to prove the Weierstrass Theorems.

Prop 0.2. *The algebra ${}_nH_0$ may be described as $\mathbb{C}\{z_1, \dots, z_n\}$ the algebra of convergent power series in n variables on some polydisc.*

Proof. See Proposition 3.1.2 of Taylor. □

Definition 0.3. Let f be a holomorphic function defined in a neighbourhood of 0, $0 \le k < \infty$.

- f has *vanishing order* k in z_n at 0 if $f(0, \dots, 0, z_n)$ has a zero of order k at 0.
- f has *finite vanishing order* in z_n at 0 if $f(0, \dots, 0, z_n)$ does not vanish identically in a neighbourhood of $z = 0$.
- The germ, $\underline{f} \in_n H_0$ has *vanishing order* k in z_n if it has a representative that has vanishing order k.

Prop 0.4. *If f is a holomorphic function in a neighbourhood U of 0 and has vanishing order k in z_n at 0 then there is a polydisc $\Delta(0, r') \times \delta(0, r_n)$ such that for each $z \in \Delta(0, r')$, $f(z', z_n)$ as a function of z_n has exactly k zeroes in $\Delta(0, r_n)$, and no zeroes on the boundary of $\Delta(0, r_n)$.*

Proof. See Proposition 3.3.1 of Taylor. □

Definition 0.5. If U is a neighbourhood of 0 and $T \subset U$ is open then T is a *thin subset* if for every $z \in U$, there exists a neighbourhood V of z with a $f \in \mathcal{H}(V)$ such that $f|_{V \cap T} = 0$, but f does not vanish identically on any neighbourhood of z.

Prop 0.6 (Removable Singularity Theorem)**.** *If f is bounded and holomorphic on an open set of the form, $U \setminus T$, where U is open and T is thin, then f has a unique holomorphic extension to U.*

Proof. See Theorem 3.3.2 of Taylor. □

Now we shall introduce elementary symmetric functions and power sum functions so we can define the Weierstrass polynomials used in the Weierstrass theorems.

Definition 0.7. An *elementary symmetric function* is of the form: $\phi_j(z)$ for $j = 1, \ldots, n$ where

$$\prod_{i=1}^{n}(\lambda - z_i) = \lambda^n - \phi(z)^{n+1}\lambda^{n+1} + \cdots + (\text{-}1)^n\phi_n(z).$$

For example: $\phi_1(z) = \sum_{i=1}^{n} z_i$ and $\phi_2(z) = \sum_{0<i<j\leq n} z_i z_j$.

Definition 0.8. A *power sum function* is of the form:

$$s_k = z_1^k + \cdots + z_n^k$$

for $k = 1, \ldots, n$.

Lemma 0.9. *Each elementary symmetric function, ϕ_j, may be written as a polynomial in power sum functions $s_1, \ldots, s_n$.*

Proof. See Lemma 3.3.3 of Taylor. □

Definition 0.10. A *Weierstrass polynomial* of degree k in z_n is a polynomial, $\underline{\underline{h}} \in_{n-1} H_0[z_n]$ of the form:

$$\underline{\underline{h}}(z) = z_n^k + a_1(z')z_n^{k-1} + \cdots + a_{k-1}(z')z_n + a_{k(z')}$$

where $z = (z', z_n)$ and a_i are non-units in ${}_{n-1}H_0$.

Now we are ready to state the first of the Weierstrass Theorems, the Weierstrass Preparation Theorem, which can be used to write a germ as the product of a unit and a Weierstrass polynomial.

Theorem 0.11 (Weierstrass Preparation Theorem)**.** *If $\underline{\underline{f}} \in_n H_0$ has vanishing order k in z_n then $\underline{\underline{f}}$ has a unique factorization as $\underline{\underline{f}} = \underline{\underline{u}}\,\underline{\underline{h}}$ where, $\underline{\underline{h}}$ is a Weierstrass polynomial of degree k in z_n is and $\underline{\underline{u}}$ is a unit in ${}_nH_0$.*

Proof. Fix a representative, f, of $\underline{\underline{f}}$. By Proposition 0.4 there is a polydisc $\Delta(0, r)$ such that $f(z', z_n)$ has exactly k zeroes, $b_1(z'), \ldots, b_1(z')$.

We want the Weierstrass polynomial:

$$h(z) = \prod_{j=1}^{k}(z_n - b_j(z')) = z_n^k - a_1(z')z_n^{k-1} + \cdots + (-1)^k a_k(z'),$$

since it has the same zeroes as $f(z', z_n)$.

We claim that $a_(z)$ are holomorphic. The $a_i(z')$ are symmetric functions of b_j's and so by the lemma, we can write $a_i(z')$ as polynomials in power sums, s_m, where

$$s_m = \sum_{j=1}^{k} b_j^m.$$

From residue theory we get:

$$s_m(z') = \frac{1}{2\pi i}\int_{|\xi|=r_m} \frac{\xi^m \frac{\partial f}{\partial \xi}(z',\xi)}{f(z',\xi)}d\xi.$$

Thus the $s_m(z')$ are holomorphic and so are the a_i's.

Since the b_j's vanish at $z=0$, the a_j's vanish at the origin. Thus the germ $\underline{\underline{h}}$ is a Weierstrass polynomial.

It remains to show that $u = f/h$ is holomorphic and non-vanishing in $\Delta(0,r)$. Then $f = uh$.

For each fixed $z' \in \Delta(0,r)$, $f/h : z_n \to \frac{f(z,z')}{h(z',z_n)}$ (ie. fix z'), has a holomorphic extension to $\Delta(0,r)$. This is because the numerator and denominator have the same zeroes and h is bounded away from 0 on the boundary of $\Delta(0,r)$. By the maximum modulus principle, f/h is bounded on $\Delta(0,r)$. Since f/h is holomorphic everywhere except $h=0$, we use the Riemann singularity Theorem to extend f/h to the whole polydisc.

The factorization is unique and therefore h and u are unique. Finding their germs we get the factorization we are looking for. □

Next we introduce the Weierstrass Division Theorem, which defines the division of germs by Weierstrass polynomials.

Theorem 0.12 (Weierstrass Division Theorem)**.** *If $\underline{\underline{h}} \in {}_{n-1}H_0[z_n]$ is a Weierstrass polynomial of degree k and $\underline{\underline{f}} \in {}_nH_0$ then $\underline{\underline{f}}$ can be uniquely written as*

$$\underline{\underline{f}} = \underline{\underline{g}}\,\underline{\underline{h}} + \underline{\underline{q}}$$

where $\underline{\underline{g}} \in {}_nH_0$ and $\underline{\underline{q}} \in {}_{n-1}H_0$ is a polynomial with degree less than k. If $\underline{\underline{f}}$ is a polynomial then so is $\underline{\underline{g}}$.

Proof. Pick representatives f and h of $\underline{\underline{f}}$ and $\underline{\underline{h}}$ respectively which are defined in a neighbourhood of the polydisc, $\Delta(0,r)$ such that $h(z',z)$ has exactly k zeroes in $\Delta(0,r_n)$ as a function of z_n for each fixed $z' \in \Delta(0,r')$, where $r = (r', r_n)$.

Define

$$g(z) := \frac{1}{2\pi i}\int_{|\xi|=r_n} \frac{f(z',\xi)}{h(z',\xi)(\xi - z_n)}d\xi.$$

Then $g(z)$ is holomorphic in $\Delta(0,r)$. Thus $q := f - gh$ is holomorphic in $\Delta(0,r)$ as well and

$$q(z) = \frac{1}{2\pi i}\int_{|\xi|=r_n} \frac{f(z',\xi)}{h(z',\xi)}\frac{h(z',\xi) - h(z'-z_n)}{\xi - z_n}d\xi.$$

However

$$\frac{h(z',\xi) - h(z',z_n)}{\xi - z_n}$$

is a polynomial in z_n with degree less than k since ξ is a z_n root of h. Therefore q is a polynomial of degree less than k.

For uniqueness, suppose that $f = gh + q = g_1 h + q_1$ are two representations with q, q_1 having degree less than k. Then $q - q_1 = (g_1 - g)h$ and thus $q = q_1$ and $g_1 = g$.

If f is a polynomial in z_n then by polynomial division, $f = gh + q$ where g and q are polynomials and q has degree less than k. By uniqueness of representation this coincides with our division above. □

We finish with a nice algebraic theorem.

Theorem 0.13. *The ring ${}_nH_0$ is Noetherian.*

Proof. Prove by induction on n.

Base Case ${}_0H_0 = \mathbb{C}$ is Noetherian.

Inductive Case Assume ${}_nH_0$ is Noetherian.

We will that show that ${}_{n+1}H_0$ is Noetherian by showing that every non-trivial proper ideal of ${}_{n+1}H_0$ is finitely generated.

Let J be a non-trivial proper ideal of ${}_{n+1}H_0$.

By the Weierstrass Preparation Theorem, there is some Weierstrass polynomial $\underline{\underline{h}}$ in J ($\underline{\underline{f}} = \underline{\underline{u}}\,\underline{\underline{h}} \in J \implies \underline{\underline{u}}^{-1}\underline{\underline{u}}\,\underline{\underline{h}} = \underline{\underline{h}} \in J$).

By definition $\underline{\underline{h}} \in {}_nH_0[z_{n+1}] \cap J$. By the inductive hypothesis ${}_nH_0$ is Noetherian. Therefore $nH_0[z_{n+1}]$ is Noetherian and thus $J \cap {}_nH_0[z_{n+1}]$ is finitely generated.

If $f \in J$, then by Weierstrass Division $\underline{\underline{f}} = \underline{\underline{g}}\,\underline{\underline{h}} + \underline{\underline{q}}$. Since $\underline{\underline{h}}, \underline{\underline{q}} \in J \cap {}_nH_0[z_{n+1}]$, $\underline{\underline{f}}$ can be gernerated by the generators of $J \cap {}_nH_0$ and thus J is finitely generated.

$\square$