Summary: Multiple Zeta Functions.

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December 3, 2012

In 1859, Riemann introduced the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is defined in the half plane $\Re(s) > 1$. He later showed that this function can be analytically continued to $s \in \mathbb{C} \setminus \{1\}$, with a pole with residue 1 at $s = 1$, and showed that $\zeta(s)$ satisfies the functional equation

$$\pi^{-s/2}\zeta(s)\Gamma(s/2) = \pi^{-\frac{1-s}{2}}\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right).$$

He also gave his famous conjecture, known as the Riemann hypothesis, that if $\zeta(s) = 0$ and $0 < \Re(s) < 1$ then $\Re(s) = 1/2$. This remains unproved.

In 1882, Hurwitz defined a "shifted" zeta function, often called the Hurwitz zeta function (HZF), defined by

$$\zeta(s;\chi) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},$$

for $0 < x \leq 1$. Note that $\zeta(s;1) = \zeta(s)$. This function was introduced to simplify the study of L functions: for any Dirichlet character $\chi(\mod q)$ we have

$$L(s;x) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = q^{-s} \sum_{a(\mod q)} \chi(a)\zeta(s;a/q).$$

The HZF also has analytic continuation (AC) to $s \in \mathbb{C} \setminus \{1\}$, with a simple pole with residue 1 at $s = 1$, and so the same holds for the L functions. Since the second argument x is always rational for the above identity, Hurwitz restricted his study to rational values in the interval $(0, 1)$.

The goal of [1] is to outline a method by which the analytic continuation (AC) of $\zeta)s;x)$ can be easily derived from the AC of $\zeta(s)$. The method they give is applicable to a more general family of multiple zeta functions (MZFs). We define two of these below.

Definition 1. We call the power series

$$\zeta(s_1, s_2, ..., s_r) = \sum_{n_1 > n_2 > ... > n_r \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} ... n_r^{s_r}}$$

the *multiple zeta function (MZF)* with *weight* $s_1 + s_2 + ... + s_r$ and *depth* r. We call the power series

$$\zeta(s_1, s_2, ..., s_r; x_1, x_2, ..., x_r) = \sum_{n_1 > n_2 > ... > n_r \geq 1} \frac{1}{(n_1 + x_1)^{s_1}(n_2 + x_2)^{s_2} ... (n_r + x_r)^{s_r}}$$

a *multiple Hurwitz zeta function (MHZF)*, with the same depth and weight.

Knowledge of special values of the RZF motivated a study of the special values of the MZF and MHZF. These are often called Multiple Zeta Values (MZVs).

Multiple zeta values.

The study of MZVs has connections to physics and several other branches of math. The theory has classical roots, beginning with Euler's result from 1734, given below. We give some others as well. It is expected among those studying these objects that the special values can be explained by a single theory. One hope is that the study of MZVs will lead to a better understanding of special values of the RZF, such as $\zeta(2k+1)$.

1734 (Euler) $2\zeta(2k) = (-1)^{k-1}(2\pi)^k \frac{B_2k}{(2k)!}$

1978 (Apry) $\zeta(3)$ irrational

2000 (Rivoal, Zudilin, Ball) Infinitely many of $\zeta(2k+1)$ are irrational.

2001 (Borwein, Bradley, Broadhurst, Lisonek) $\frac{\zeta(3,1,3,1,\ldots,3,1)}{2n} = \frac{2\pi^{4n}}{(4n+2)!}$

Recent (Zudilin) One of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational and for some $j \in [5, 69]$, the set $\{1, \zeta(3), \zeta(j)\}$ is linearly independent over $\mathbb{Q}$.

Analytic continuation

We turn back to the analytic continuation of these functions. We begin with a general theorem.

Theorem 1. *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series which is absolutely convergent in $\Re(s) > 1$. *Suppose that* $f(s)$ *extends to a meromorphic function* $\forall s \in \mathbb{C}$. *Then*

$$f(s;x) = \sum_{n=1}^{\infty} \frac{a_n}{(n+x)^s}$$

also extends meromorphically for all $s \in \mathbb{C}$. *If* $f(s)$ *has a pole at* $s = 1$ *with residue 1, then so does* $f(s;x)$.

It is immediately obvious that this applies to our RZF and HZF.

Proof. We may assume that $0 < |x| < 1$, otherwise we begin summation at some n_0 with $n_0 > |x|$. For $\Re(s) > 1$,

$$\begin{aligned} f(s;x) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s}(1+x/n)^{-s} \\ &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \sum_{r=0}^{\infty} \binom{-s}{r} (x/n)^r \\ &= \sum_{r=0}^{\infty} \binom{-s}{r} \sum_{n=1}^{\infty} \frac{a_n}{n^{s+r}} x^r \\ &= \sum_{r=0}^{\infty} \binom{-s}{r} f(s+r) x^r. \end{aligned}$$

For sufficiently large r, $f(s+r)$ is bounded. Applying the rth root test will show that the series $f(s;x)$ converges absolutely for $|x| < 1$ and $\Re(s) > 1$ (see [1] for details). If we increase the lower bound of summation to $r = 1$, we find that the function is absolutely convergent on $\Re(s) > 0$. If we increase the lower bound to $r = M+1$, the function is absolutely convergent for $\Re(s) > -M$. Since the function is meromorphic for $r = 1 \ldots M$, we deduce that the whole function is then meromorphic for $\Re(s) > -M$, and since M is arbitrary, we have the proof. The poles and residue carry from integer translates of the poles and residues of $f(s)$. □

The method of extracting a binomal series in the above proof is called the binomial principle of analytic continuation. This is applied with great success in [1] to $\zeta(s)$, $\zeta(s;x)$ and $\zeta(s_1,s_2,...,s_r;x_1,x_2,...,x_r)$. Applying this method to $\zeta(s)$ gives the following theorem.

Theorem 2.

$$-\frac{1}{x^s}+\zeta(s;x)-\zeta(s)=\sum_{r=1}^{\infty}\binom{-s}{r}\zeta(s+r)x^r$$

This identity gives the AC of $\zeta(s;x)$ in an inductive way, once we know the AC of $\zeta(s)$. By taking $x=\frac{1}{2}$ in $\zeta(s;x)$, and doing a small amount of work, we arrive at the next theorem.

Theorem 3.

$$(2^s-2)\zeta(s)=2^s+\sum_{r=1}^{\infty}\binom{-s}{r}2^{-r}\zeta(s+r)$$

For $\Re(s)>0$, the right hand side of the above identity is analytic. Thus, $\zeta(s)$ as a meromorphic continuation to $\Re(s)>0$ with possible poles at

$$s=1+\frac{2\pi im}{\log 2},\quad m\in\mathbb{Z}.$$

The following theorem is instrumental in finding the complete AC.

Theorem 4. *For $q\in\mathbb{N}$,*

$$(q^s-q)\zeta(s)=\sum_{a=1}^{q}\left[\zeta(s;a/q)-\zeta(s)\right].$$

Using Theorem 2, we get an AC of $\zeta(s;a/q)-\zeta(s)$ for $\Re(s)>0$, with possible poles at

$$s=1+\frac{2\pi im}{\log 2},\quad m\in\mathbb{Z}.$$

Taking $q=3$, we find that $\zeta(s)$ extends analytically in the same region, with the only pole at $s=1$. Applying Theorem 3 will show the residue is 1 at $s=1$, and induction gives:

Theorem 5. *$\zeta(s)$ extends analytically $\forall s\in\mathbb{C}\setminus\{1\}$.*

By using Theorems 2 and 1, we get:

Theorem 6. *$\zeta(s;x)$ extends analytically $\forall s\in\mathbb{C}\setminus\{1\}$.*

The question now is how to extend this to the MHZF? If we fix $s_2,...,s_r$, then we can apply the binomial principle to extend the resulting single variable function to $\mathbb{C}$, but we would rather extend the whole function in one argument. The result that makes this possible is the following.

Theorem 7. *(Hartog's Theorem) Let f be a function defined on $\mathbb{C}^{r\geq 2}$, and fix $r-1$ variables. If the resulting function is analytic in some open set $D\subset\mathbb{C}^r$, then the function f itself is analytic in D.*

Theorem 8. *(Hartog's Phenomenon) If f is a holomorphic function on a set $G\setminus K$, where G is an open subset of $\mathbb{C}^{r\geq 2}$ and K is a compact subset of G such that $G\setminus K$ is connected, then f extends to a holomorphic function on G.*

This is only true for $r\geq 2$. For a counter-example, take $f(z)=1/z$, $K=\{0\}$ and $G=\mathbb{C}$.

References

[1] M. Murty and K. Sinha. Multiple hurwitz zeta functions. *Proceedings of Symposia in Pure Mathematics*, 75, 2006.