Summary: Analytic Nullstellensatz Pt 1.

Sam Johnson

19 November, 2012

This week's seminar lecture covers the first half of the material on the analytic version of the Nullstellensatz used in the analytic geometry associated with the analysis of several complex variables. We introduce some necessary definitions from analytic geometry first, and then give a road-map of the proof, moving through 4 main steps, comparing the arguments to those needed for Hilbert's Nullstellensatz from general algebraic geometry. The details of these four main steps will be largely omitted, left to the second part of the lecture.

Definitions for varieties

Definition 1. Let $U \subset \mathbb{C}^n$ be open, $S \subset \mathcal{H}(U)$, $|S| < \infty$. Then

$$V(S) = \{z \in U : f(z) = 0 \ \ \forall f \in S\}.$$

For a closed $T \subset U$, we say that T is a *holomorphic subvariety* of U if for every $\lambda \in U$ there exists a neighbourhood $W_\lambda \ni \lambda$ and a finite set $S_\lambda \in \mathcal{H}(W_\lambda)$ such that

$$T \cap W_\lambda = V(S_\lambda).$$

Note: we are defining these varieties analogously to general algebraic varieties. We have replaced regular functions with holomorphic ones, and Tsariski open sets with open sets in the Euclidean topology.

Example 1. Let

$$\begin{aligned} N_k &= \{z = (z_1, ..., z_n) \in \mathbb{C}^n \ : \ z_{k+1} = z_{k+2} = ... = z_n = 0, \ Re(z_k) > 0\}, \\ H_k &= \{z \in \mathbb{C}^n \ : \ Re(z_k) > 0\}. \end{aligned}$$

Then N_k is a holomorphic subvariety of V_k but not of $\mathbb{C}^n$, since it is not closed in $\mathbb{C}^n$, but N_k is not an algebraic subvariety of V_k because V_k is not Tsariski open.

Definition 2. Two varieties V, W on U are equivalent at $\lambda \in V \cap W$ if there is some neighbourhood $U_\lambda \ni \lambda$ such that $V \cap U_\lambda = W \cap U_\lambda$.

The germ of V at $\lambda \in V$ is the equivalence class containing V, we write $\underline{V}$ to denote the germ of V.

In order to state the Nullstellensatz, we need to define the ideal and locus of a variety.

Definition 3. Let $\underline{V}$ be the germ of holomorphic variety at 0. Then $id(\underline{V})$ is the ideal of ${}_n\mathcal{H}_0$ of functions vanishing on $\underline{V}$.

Definition 4. Let I be an ideal of ${}_n\mathcal{H}_0$ and S be a finite generating set generating I (this exists becuase ${}_n\mathcal{H}_0$ is Noetherian). Then $loc(I)$ is the germ of $V(S)$.

We omit the proof of the following.

Proposition 1. *Let V be a holomorphic variety at* 0*, and I be an ideal of ${}_n\mathcal{H}_0$. Then*

1. $\underline{V_1} \subset \underline{V_2} \Rightarrow id\underline{V_1} \supset id\underline{V_2}$,
2. $I_1 \subset I_2 \Rightarrow locI_1 \supset locI_2$,
3. $\underline{V} = loc(id\underline{V})$,
4. $I \subset id(locI)$.

Definition 5. Let I be an ideal of a ring R. We define the radical of I to be

$$\sqrt{I} = \{f \in R \ : \ f^k \in I \text{ for some k }\}.$$

The Nullstellensatz

Our goal is to show

$$id(locI) = \sqrt{I}$$

for:

1. I an ideal of $\mathbb{C}[z_1, ..., z_n]$ (Hilbert's Nullstellensatz);
2. I an ideal of ${}_n\mathcal{H}_0$ (Rückert's analytic Nullstellensatz).

Note that in both cases, $\sqrt{I} \subset id(locI)$ is clear. The hard part is in proving the opposite inclusion. The proof of both cases has four main results to stand on. We first outline these below of Hilbert's Nullstellensatz.

1. **Reduce to prime ideals.** Let A be a Noetherian ring. Then if $id(locI) = \sqrt{I}$ for prime ideals, then $id(locI) = \sqrt{I}$ for all ideals.
2. **Noether normalisation.** Let A be a finitely generated commutative algebra over an infinite field K. Then there is an algebraically independent set $\{x_1, ..., x_n\} \subset A$ consisting of linear combinations of the generators of A such that if $B = K[x_1, ..., x_n]$ then A is a finite extension of B.
3. **[1, Cor. 4.3.3].** Let A be an integral extension of B, where A and B are integral domains. Let P be a prime ideal of A. Then P is maximal if and only if $P \cap B$ is a maximal ideal of B.
4. **Going up theorem.** Let A be an integral extension of B and P be a prime ideal of B. Then $P = Q \cap B$ for some prime ideal Q of A.

Assuming steps 1 through 4 above allow a proof of Hilbert's Nullstellensatz. Below, we outline the analogous steps for the analytic version.

1. **Reduce to prime ideals**
2. **[1, Theorem 4.5.2]** Let P be a non-zero prime ideal of ${}_n\mathcal{H}_0$. Then we can choose linear coordinates for $\mathbb{C}^n$ and $m < n$ such that $A = {}_n\mathcal{H}_0/P$ is a finite extension of $B = {}_m\mathcal{H}_0$ generated by the images of $z_{m+1}, ..., z_n$.

3 & 4 **[1, Theorem 4.5.4]** Let P be a prime ideal of ${}_n\mathcal{H}_0$. Suppose that the coordinates for C_n and $m < n$ are chosen so that ${}_n\mathcal{H}_0/P$ is a finite extension of ${}_m\mathcal{H}_0$. Then $\underline{V} = locP$ is a union $\underline{V} = \underline{V}' \cup \underline{V}''$ of germs such that $\pi : \mathbb{C}^n \to \mathbb{C}^m$ makes $\underline{V}'$ the germ of a finite branched holomorphic cover of pure order on a neighbourhood of $0 \in \mathbb{C}^m$.

Note that the phrase *finite branched holomorphic cover of pure order* has not yet been defined. This is defined in part 2 of the lecture.

Hilbert's Nullstellensatz

By the reduction to prime ideals, we may assume that I is prime. Recall also that we need only show that $id(locI) \subset \sqrt{I}$.

Take $f \in \mathbb{C}[z_1, ..., z_n]$, $f \notin I$. By Noether normalisation, we can choose coordinates and $m < n$ such that $A = \mathbb{C}[z_1, ..., z_n]/I$ is a finite extension of $B = \mathbb{C}[z_1, ..., z_m]$. That is, the image of f in A satisfies a minimal polynomial over B of the form

$$f^k + b_{k-1}f^{k-1} + ... + b_1 f + b_0 \in I,$$

with $b_i \in B$. By the minimality of this polynomial, and the fact that I is a prime ideal, we know that b_0 is not identically 0. Thus, there exists a point $\lambda \in \mathbb{C}^m$ with $b_0(\lambda) \neq 0$. Let $M = id\{\lambda\}$, the ideal of polynomials in B which are zero at λ. This is a maximal ideal of B, that is, $B/M = \mathbb{C}$.

By the going up theorem, there must then be a prime ideal N of A such that $N \cap B = M$, and by [1, Cor 4.3.3], N is maximal in A. Furthermore, A/N is a finite extension of B/M. But $\mathbb{C}$ is algebraically closed, so $A/N = \mathbb{C}$. Let $\zeta \in \mathbb{C}^n$ be the point such that ζ_i is the image of z_i in A/N. Then $N = id\{\zeta\}$. (Note: $\zeta_i = \lambda_i$ for $i \leq m$, since $N \cap B = M$.)

Now $A = \mathbb{C}[z_1, ..., z_m]/I$, so all polynomials in I must vanish at ζ, including the minimal polynomial over B satisfied by f:

$$f^k(\zeta) + b_{k-1}(\lambda)f^{k-1}(\zeta) + ... + b_1(\lambda)f(\zeta) + \underbrace{b_0(\lambda)}_{\neq 0} = 0.$$

That is, $f(\zeta) \neq 0$, so $f \notin id(locI)$. Thus, $id(locI) \subset I \subset \sqrt{I}$, completing the proof.

Analytic Nullstellensatz

Again, we may assume that I is a prime ideal, and we need only show that $id(locI) \subset I$. The proof will largely follow the same outline as the Hilbert Nullstellensatz.

Take $f \in {}_n\mathcal{H}_0$, $f \notin I$. By [1, Thm 4.5.2], we may choose coordinates and $m < n$ such that $A = {}_n\mathcal{H}_0/I$ is a finite extension of $B = {}_m\mathcal{H}_0$. As before, the image of f in A satisfies a minimal polynomial over B of the form

$$f^k + b_{k-1}f^{k-1} + ... + b_1 f + b_0 \in I,$$

where the $b_i \in {}_m\mathcal{H}_0$. By minimality of the polynomial and primality of I, we have $b_0 \neq 0$. Choose a polydisc $\Delta \in \mathbb{C}^m$ where each b_i has a representative, and let $\Pi : \mathbb{C}^n \to \mathbb{C}^m$ be the natural projection. By [1, Thm 4.5.4], after shrinking Δ if necessary, we find that $locI$ has a representative V in $\Pi^{-1}(\Delta)$ and

$$V = V' \cup V'',$$

where $\Pi : V' \to \Delta$ is a finite branched holomorphic cover of pure order.

Let f be a representative of $\underline{f}$ on a neighbourhood U of 0, with $U \subset \Pi^{-1}(\Delta)$. In the next part of this lecture, we will find that U contains points $z \in V' \subset locI$ arbitrarily close to 0 with

$$f^k(z) + a_{k-1}(\Pi(z))f^{k-1}(z) + ... + \underbrace{a_0(\Pi(z))}_{\neq 0} = 0.$$

Thus $f(z) \neq 0$, so $\underline{f}$ does not vanish on $\underline{V}$ and $f \notin id(locI)$, giving the result.

Conclusion

This finishes both proofs, under some fairly heavy assumptions. In the second part, many details of these assumptions are filled in, and the mysterious *finite branched holomorphic cover of pure order* is defined.

References

[1] J. Taylor. *Several Complex Variables with Connections to Algebraic Geometry and Lie Groups.* AMS Graduate Studies in Mathematics, 2002.