Strict minimal points via surgery

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We follow [1, Section 9.2], with some restrictions:

- we work in $\mathbb{C}^2$ (since if we can generalise to $d = 2$, we can generalise to any finite dimension);
- we assume $F = G/H$ is rational, ie $F \in \mathbb{C}(x, y)$.

Then the Cauchy integral formula becomes

$$A_{rs} = \frac{1}{(2\pi i)^2} \int_T \frac{F(x,y)}{x^{r+1}y^{s+1}} dxdy$$

where $T = \{(x, y) \in \mathbb{C}^2 \ : \ |x| = \epsilon_1, |y| = \epsilon_2\}$ for ϵ_1, ϵ_2 sufficiently small.

Theorem 1. *Let $F = G/H \in \mathbb{C}(x, y)$ and fix $\Delta_* = (r_*, s_*) \in \mathbb{R}^2_+$. Assume that $h_* : \mathbb{V} \to RR$, the height function on the singular variety of F, has a unique minimum (x_0, y_0) which is a smooth point. Then there is $\mathbb{D}_1$ (resp $\mathbb{D}_2$) a disc of $\mathbb{C}$ centred at x_0 (resp y_0) such that*

$$a_{rs} \sim f_*(r, s) = \frac{1}{(2\pi i)^2} \int_{\mathcal{N} \times \mathbb{D}_2} \frac{F(x,y)}{x^{r+1}y^{s+1}} dxdy,$$

where $\mathcal{N} = \mathbb{D}_1 \int \{x \in \mathbb{C} \mid |x| = |x_0|\}$. Moreover, there is a holomorphic function $\phi : \mathbb{D}_1 \to \mathbb{D}_2$ such that

$$f_*(r, s) = \frac{1}{2\pi i} \int_{\mathcal{N}} \frac{1}{x^{r+1}\phi(x)^s} Res\left\{\frac{F(x,y)}{y}, \ y = \phi(x)\right\} dx.$$

Definition 1. *Let $H \in \mathbb{C}[x, y]$ and $\mathbb{V} = \{(x, y) \in \mathbb{C}^2 \mid H(x, y) = 0\}$. A point $(x_0, y_0) \in \mathbb{V}$ is smooth if $\nabla H|_{(x_0,y_0)} \neq 0$.*

Example 2. *There are two examples.*
1. *Let $H(x, y) = 1 - x - y$. Then $\partial_x H = -1$ and $\partial_y H = -1$, so all points of H are smooth. From another point of view, H is flat everywhere (which we can see part of by looking at the section of $H(x, y)$ in $\mathbb{R}$, the red line in Figure **??**).*

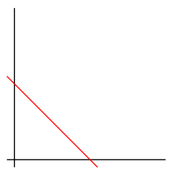

Figure 1: The red line is the real portion of the variety of $H(x, y) = 1 - x - y$, which is globally flat

2. *Let $H(x, y) = (1 + x)x^2 + y^2$. The real variety of H is shown in Figure **??**. The non-smooth point is $(0, 0)$, which is where the curve intersects itself. We could see this in two ways: 1) $(0, 0)$ is a double point, so $\partial_x(H)|_{(0,0)} = \partial_y(H)|_{(0,0)} = 0$; 2) $\partial_y(H) = 2y$, which is only zero if $y = 0$, and $(0, 0)$ is the only point of $\mathbb{V}(H)$ satisfying this.*

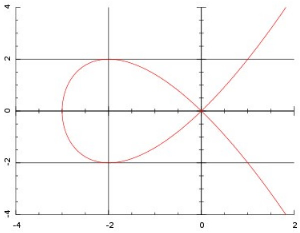

Figure 2: The algebraic curve $H(x,y) = (1+x)x^2 + y^2$

Theorem 3. *Let $\mathbb{V} = \{(x,y) \in \mathbb{C}^2 \mid H(x,y) = 0\}$ be an algebraic curve and $(x_0, y_0) \in \mathbb{V}$ be a smooth point. Then there is a $\mathbb{D}_1$ (resp $\mathbb{D}_2$), a disc of $\mathbb{C}$ centred at x_0 (resp y_0) and a holomorphic function $\phi : \mathbb{D}_1 \to \mathbb{D}_2$ such that*

$$\mathbb{V} \cap (\mathbb{D}_1 \times \mathbb{D}_2) = \{(x, \phi(x) \mid x \in \mathbb{D}_1\}.$$

Proof. First, (x_0, y_0) is smooth, so at least one of $\partial_x H|_{(x_0,y_0)}$ and $\partial_y H|_{(x_0,y_0)}$ is non-zero. Without loss of generality, we may assume that $\partial_h H|_{(x_0,y_0)} \neq 0$.

Now, assume that there exists a function $f : \mathcal{D} \to \mathbb{C}$ with $\forall z \in \partial\mathcal{D}$, $f(z) \neq 0$. Then the claim is that

$$\# \text{ of zeroes of } f \text{ in } \mathcal{D} = \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{\partial_z f(z)}{f(z)} dz.$$

Indeed, if f has k roots in $\mathcal{D}$, then

$$f(z) = (z - z_1)^{\alpha_1} (z - z_2)^{\alpha_2} ... (z - z_k)^{\alpha_k} \tilde{f}(z),$$

where $\tilde{f}$ is non-zero on $\mathcal{D}$. Taking the first derivative, we find

$$d_z f = \sum_{i=1}^{k} \left[\alpha_i (z - z_1)^{\alpha_1} ... (z - z_i)^{\alpha_i - 1} ... (z - z_k)^{\alpha_k} \tilde{f}(z) + (z - z_1)^{\alpha_1} ... (z - z_k)^{\alpha_k} dz \tilde{f}(z) \right].$$

Thus,

$$\frac{d_z f}{f} = \sum_{i=1}^{k} \left[\frac{\alpha_i}{z - z_i} + \frac{d_z \tilde{f}(z)}{\tilde{f}(z)} \right].$$

Taking the integral of this, we take the residue of each summand at z_i. Since $\tilde{f}$ is non-zero on $\mathcal{D}$, the second part of each summand is integrated to zero, and we get

$$\frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{\partial_z f(z)}{f(z)} dz = \sum_{i=1}^{k} \alpha_i + 0,$$

which is the number of zeroes, with multiplicity.

From this, if f has a unique zero in $\mathcal{D}$, z_0 say, then the modified integral will allow us to find it:

$$\frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{z \partial_z f(z)}{f(z)} dz = z_0.$$

Now, fix $x = x_0$ and $H(x_0, y) = H_{x_0}(y)$. We then have $H_{x_0}(y_0) = 0$ and $\partial_y H|_{(x_0,y_0)}$. Thus H_{x_0} is not flat at y_0, so there is a neighbourhood of y_0 in which $H_{x_0} \neq 0$. Take this neighbourhood to be $\mathbb{D}_2$. By our previous claim, we know that

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}_2} \frac{\partial_y H_{x_0}(y)}{H_{x_0}(y)} dy = 1 \Rightarrow \frac{1}{2\pi i} \int_{\partial\mathbb{D}_2} \frac{y \partial_y H_{x_0}(y)}{H_{x_0}(y)} dy = y_0. \tag{1}$$

Further, $H \in CC[x,y]$ tells us that $H_x(y)$ depends continuously on x.

Now, the function $H_{x_0}(y)$ is non-zero for all $y \in \partial\mathbb{D}_2$. By continuity, there must be some neighbourhood of x_0, call it $\mathbb{D}_1$, centred at x_0 such that $\forall x \in \mathbb{D}_1$, $H_x(y) \neq 0\ \forall y \in \mathbb{D}_2$.

So we can replace x_0 in Statement 1 by any $x \in \mathbb{D}_1$, giving

$$\phi(x) = \int_{\partial\mathbb{D}_2} \frac{y d_y H_x(y)}{H_x(y)} dy = y \in \mathbb{D}_2.$$

□

Fix $\Delta_* = (r_*, s_*) \in \mathbb{R}^2_+$. Then there is an associated height function on the singular variety of F, or on its amoeba

$$h_* : \mathbb{V} \to \mathbb{R}, \qquad (x, y) \mapsto -\langle \Delta_*, (\log|x|, \log|y|)\rangle = -r_* \log|x| - s_* \log|y,$$
$$h_* : Re \log \mathbb{V} \to \mathbb{R}, \qquad (x, y) \mapsto -r_* \log|x| - s_* \log|y.$$

Recall that for a function F, amoeba$(F) = Re \log \mathbb{V}$, where $\mathbb{V}$ is the singular variety of F. Then the components B of $\mathbb{R}^2 \setminus Re \log \mathbb{V}$ are the portions of $\mathbb{R}^2$ in which F has a Laurent series representation, and F will have a minimum on ∂B.

Lemma 4. *Let $h_* : Re \log \mathbb{V} \to \mathbb{R}$ be as above. Then h_* takes its extremal values on $\partial Re \log \mathbb{V}$.*

Observation 1. *We observe that*

$$a_{rs} \underset{\substack{r+s\to\infty \\ (r,s)||\Delta_*}}{\sim} f_*(r, s)$$

if and only if $x_0^r y_0^s [a_{rs} - f_*(r, s)] = o(1)$ for $r + s \to \infty$ and $r/s = r_*/s_*$.

Theorem 5 (Restatement of Theorem 1)**.** *Let $F = G/H \in \mathbb{C}(x, y)$ and $\Delta_* = (r_*, s_*) \in \mathbb{R}^2_+$. Assume that $h_* : \mathbb{V} \to \mathbb{R}$ has a unique critical point (x_0, y_0) that is smooth. Then there is a disc $\mathbb{D}_1$ (resp $\mathbb{D}_2$) of $\mathbb{C}$, centred at x_0 (resp y_0) such that*

$$a_{rs} \sim f_*(r, s) = \frac{1}{(2\pi i)^2} \int_{\mathcal{N}\times\mathbb{D}_2} \frac{F(x, y)}{x^{r+1} y^{s+1}} dx dy.$$

Proof. Since (x_0, y_0) is a smooth point, at least one of $\partial_x H|_{(x_0,y_0)}$ and $\partial_y H|_{(x_0,y_0)}$ is non-zero. We pick the y coordinate.

From the Cauchy formula, we know (substituting ω for the integrand)

$$a_{rs} = \frac{1}{(2\pi i)^2} \int_T \omega,$$

where T is a torus passing through x_0. From our previous results, we know that we have the discs $\mathbb{D}_1$ and $\mathbb{D}_2$ centred at x_0 and y_0 respectively. Let δ_2 be the radius of $\mathbb{D}_2$.

In Figure 3, we can see $\mathbb{D}_1$ and $\mathbb{D}_2$ marked in red. In the x-plane, the torus T is the black circle. In the y-plane, $\mathcal{C}_- = \{y \in \mathbb{C} \ : \ |y| = |y_0| - \delta_2\}$ and $\mathcal{C}_+ = \{y \in \mathbb{C} \ : \ |y| = |y_0| + \delta_2\}$, both with positive orientation in the anti-clockwise direction. Using these two new contours, $\mathcal{C}_\pm$, and the fact that (x_0, y_0) is the unique critical point of h_*, we can rewrite the Cauchy integral as

$$a_{rs} = \frac{1}{(2i\pi)^2} \int_{|x|=|x_0|} x^{-r-1} \left[\int_{\mathcal{C}_-} - \int_{\mathcal{C}_+} \right] \frac{F(x, y)}{y^{s+1}} dy dx,$$

where we abuse notation between the square brackets, meaning the difference of the integrals over $\mathcal{C}_+$ and $\mathcal{C}_-$. Applying Observation 1, we want to show that

$$x_0^r y_0^s \left[a_{rs} - \frac{1}{(2i\pi)^2} \int_{|x|=|x_0|} x^{-r-1} \left[\int_{\mathcal{C}_-} - \int_{\mathcal{C}_+} \right] \frac{F(x, y)}{y^{s+1}} dy dx \right] = o(1).$$

The key observation is that in

$$\frac{1}{(2i\pi)^2} \int_{|x|=|x_0|} x^{-r-1} \int_{\mathcal{C}_-} \frac{F(x, y)}{y^{s+1}} dy dx$$

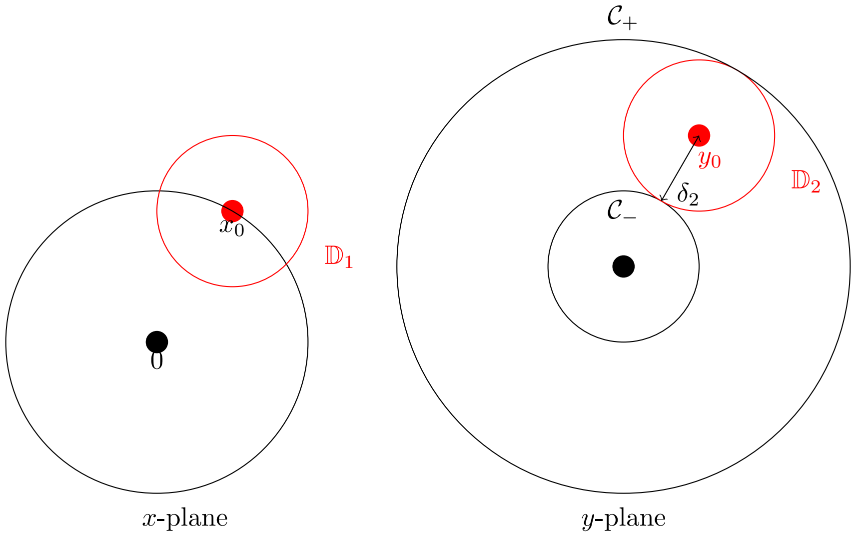

Figure 3: A pictorial representation of the integration contours in the proof of Theorem 5.

the inner integral is exponentially small away from x_0. This is due to the fact that the radius of convergence for $x \neq x_0$ is greater than $|y_0|$, since (x_0, y_0) is the unique minimal point of the singular variety. This gives

$$\left| \int_{\mathcal{C}_-} \frac{F(x,y)}{y^{s+1}} dy \right| \leq \frac{C(x)}{(|y_0| + \epsilon)^s},$$

for some $\epsilon > 0$ and x away from x_0. Similarly, the integral over $\mathcal{C}_+$ is bounded by the same quantity (up to a factor dependent on x) for x away from x_0.

By substituting this bound into the original integral, and taking a compact $K \subset \{|x| = |x_0|\}$ such that $x_0 \notin K$, we find

$$\left| \int_K \int_{\mathcal{C}_\pm} \frac{F(x,y)}{x^{r+1} y^{s+1}} dx dy \right| \quad \leq \quad \frac{C_K}{|x_0|^r (|y_0| + \epsilon)^s}.$$

We may use a single ϵ, since by the continuity of the radius of convergence, one exists for all compact $K \subset \{|x| = |x_0|\}$.

This computation works since r, s both go to infinity. If one were to remain finite while the other diverged, this would no longer hold. Multiplying by $x_0^r y_0^s$, we obtain an expression which is exponentially small.

$$\left| x_0^r y_0^s \int_K \int_{\mathcal{C}_\pm} \frac{F(x,y)}{x^{r+1} y^{s+1}} dx dy \right| \quad \leq \quad C_k \left(\frac{|y_0|}{|y_0| + \epsilon} \right)^s.$$

Thus, the contribution to the iterated integral from the compact subset K of T in the x-plane is negligible, and the asymptotic estimate is given by the integral over the product $\mathcal{N} \times \partial \mathbb{D}_2$. $\square$

References

[1] R. Pemantle and M. C. Wilson. *Analytic Combinatorics in Several Variables (draft)*. Cambridge University Press, 2012.