ASSIGNMENT 1 – SOLUTIONS

MATH 303, FALL 2011

If you find any typos or other errors please let me know.

Manipulation

(M1)

$$\begin{aligned}\{\{\{\emptyset\},\{\emptyset\}\},\{\emptyset,\{\emptyset\}\},\{\{\emptyset\}\}\} &= \{\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\},\{\{\emptyset\}\}\}\\ &= \{\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}\end{aligned}$$

(M2) *There are a number of possible approaches, here is one way* Assume $A \subseteq B$. We want to show $A \cup B = B$. It is always the case that $B \subseteq A \cup B$, so if we can show $A \cup B \subseteq B$ then we will be done. Lets try to do that. Take any element $x \in A \cup B$. Then we know that $x \in A$ or $x \in B$. In the first case we have $x \in A \subseteq B$, and so $x \in B$. Thus either way $x \in B$, and so $A \cup B \subseteq B$ which is what we needed to show.

(M3) *There are a number of possible approaches, here is one way* Supposse $A \cup B = B$. Take any $x \in A$. Then $x \in A \cup B$, so $x \in B$. Thus $A \subseteq B$.

(M4) $\bigcup \mathcal{C} = \{\emptyset\} \cup \{\emptyset, \{\emptyset\}\} \cup \emptyset = \{\emptyset, \{\emptyset\}\}$.

(M5) **(1 point)**

- To start we have $\emptyset$.
- Pair $\emptyset$ with $\emptyset$ to get $\{\emptyset, \emptyset\} = \{\emptyset\}$.
- Pair $\{\emptyset\}$ with itself to get $\{\{\emptyset\}\}$.
- Pair $\{\emptyset\}$ with $\emptyset$ to get $\{\emptyset, \{\emptyset\}\}$.
- Pair $\{\emptyset, \{\emptyset\}\}$ with itself to get $\{\{\emptyset, \{\emptyset\}\}\}$.
- Pair $\{\emptyset\}$ with $\{\{\emptyset\}\}$ to get $\{\{\emptyset\}, \{\{\emptyset\}\}\}$.
- Then $\{\{\emptyset\}, \{\{\emptyset\}\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

Pure Math

(P1) One approach which isn't too messy is to suppose there is a minimal counterexample and then show this is impossible. To do this we need to know what "minimal" should mean. One way is just to count how many times we used paring in forming a given set.

Suppose that A and B were formed differently out of $\emptyset$ and pairing, but are the same set. Suppose furthermore, that the total number of times we used pairing in forming both A and B is minimal among such examples.

Certainly since $A = B$, then A and B have the same number of elements. There are two cases.

If A and B each have one element, then write $A = \{a\}$ and $B = \{b\}$. By the axiom of extension, $a = b$. But since $A = B$ was the minimal example of sets which were equal despite being formed differently from $\emptyset$ and paring, we must have that a and

b were formed the same way. Thus A and B were formed the same way, which is a contradiction.

If A and B each have two elements, then write $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. By the axiom of extension, ($a_1 = b_1$ and $a_2 = b_2$) or ($a_1 = b_2$ and $a_2 = b_1$). By reordering if necessary we can say $a_1 = b_1$ and $a_2 = b_2$. But as in the previous case since $A = B$ was minimal we have that a_1 and b_1 were formed the same way and a_2 and b_2 were formed in the same way. Thus A and B were formed in the same way which is a contradiction.

Taking all this together we get that a minimal counterexample is impossible, thus there is no counterexample.

(P2) It is not possible to build an infinite set with only $\emptyset$, paring and unions using a finite number of steps.

To see this note that paring can only create sets with 1 or 2 elements. $\bigcup \mathcal{C}$ can only create infinite sets in two cases, first if $\mathcal{C}$ is infinite, and second if one of the elements of $\mathcal{C}$ is infinite. In both cases we had an infinite set beforehand. Thus we cannot generate an infinte set without having one first or without using an infinite number of steps.

I intended "possible" to mean "possible in a finite number of steps" (can you actually do *it if it takes infinitely many steps!), but I didn't say this, so the other answer is also acceptable provided you justify it*

Ideas

These questions are more individual, so there is less I can usefully say as a "solution". In both cases I am looking for you to discuss an example of the same sort as the barber, or the list that lists lists..., etc.