Math 303, Fall 2011, Lecture 10

1. Continue the presentation.

   I'll present the two that no group chose

   Next lets think about which of them seem intuitively plausible and which don't

$\bigtimes_{i \in I} Y_i \neq \emptyset$ — intuitive

Choice sets/functions — close

Well ordering — not intuitive

Zorn's Lemma — close

$A \leftrightarrow A \times A$ intuitive

De Bruijn - Erdős no way!

Banach-Tarski no!

$\boxed{\text{A subset of } \mathbb{R} \text{ with no countable subset} \quad \text{Sequentially continuous does not imply continuous}}$

A subset of $\mathbb{R}$ with no countable subset no!

Sequentially continuous does not imply continuous no!

every vector space has a basis who knows.

$\boxed{\text{Vector space bases}}$

$\mathbb{R}^n$ $\quad \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is a vector in $\mathbb{R}^n$ $\leftarrow$ The canonical example

other examples $\mathbb{P}_n$ polynomials of deg $\leq n$

$\mathbb{P}$ all polynomials (no degree restriction)

$\underline{\mathcal{C}[a,b]}$ continuous functions on the interval $[a,b]$

All vector spaces

Def linearly independent. A set of vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k$ is linearly independent if

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_k\underline{v}_k = \underline{0}$$

implies $c_1 = c_2 = \dots = c_k = 0$

eg $1\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} - 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

so $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is not lin. ind.

Def span. A set of vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k$ spans the vector space V if any $\underline{v} \in V$ can be written

$$c_1\underline{v}_1 + \dots + c_k\underline{v}_k = \underline{v}$$

eg $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ spans $\mathbb{R}^2$

to see this take $\begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{R}^2$

$$0\begin{bmatrix}1\\2\end{bmatrix} + a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

Def a basis is a set of vectors which is both linearly indep and spans eg $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

What about infinite dim'l vector spaces?

Answer

With the axiom of choice

→ same game. Every vector space has a basis...

Without you can have vector spaces with no basis.

if you assume the negation of AC always get this.

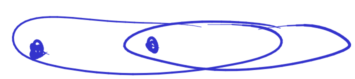
One more: Tarski's Thm.

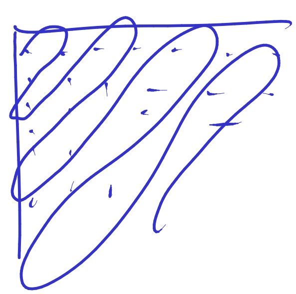
Something else equiv. to the axiom of choice:

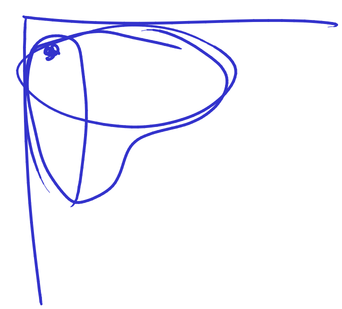
Let A be an infinite set

there is a one-to-one and onto map between A and $A \times A$.

(if A is well ordered







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② What shall we do next

There are 3 places we could go next in the course

① Go through Stromberg's proof of the Banach-Tarski paradox

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Main ref - Stromberg paper linked from website
Background requirement - calculus, rotation matrices
Feel - technical but what a cool result.

② Develop ordinals and cardinals

Main ref - Halmos

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Background requirements - what we already did

Feel - bigger infinities, Halmos style.

Joy of sets

③ Underpin what we've already done with logic

Main ref - notes, Cohen

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Background requirements - none
Feel - formal

Partial orders

doesn't have to be usual $\leq$

let P be a set. let $\leq$ be a relation on P

(ie for any $a, b \in P$

$a \leq b$? true or false)

Satisfying ① $a \leq a$ for all $a \in P$

② if $a \leq b$, $b \leq c$ then $a \leq c$

③ if $a \leq b$ and $b \leq a$ then $b = a$

eg Take $\{1, 2, \ldots\}$ set of positive integers

define a partial order using divisibility

read $2|4$ as "2 divides 4"

check ① does $a|a$ for all a? yes because $a = 1a$ so $a|a$

② If $a|b$ and $b|c$ does $a|c$?

yes $a|b$ means there is a q such that $aq=b$

$b|c$ means there is a r such that $br=c$

so pluggin into $br=c$ get
$$a(qr)=c$$
so $a|c$

③ If $a|b$ and $b|a$ does $a=b$?

yes because

$a|b$ means there is a q such that
$$aq=b$$
$b|a$ means there is an r such that
$$br=a$$

so subbing in $brq=b$

$b\neq 0$ so $rq=1$

and so since our set is $\{1,2,\ldots\}$

$r = q = 1$

so $a = b$

but partial orders have one way in which they're differ~~ent~~ from usual orders

ie $3 \nmid 2$ also $2 \nmid 3$

If this doesn't happen call it a total order

Ie a total order on a set P is a partial order $\leq$ with the extra property

(4) for all $a, b \in P$ either $a \leq b$ or $b \leq a$

Finally a well order on a set P is a total order $\leq$ with the extra property

(5) for every nonempty subset $Q \subseteq P$ Q has a least element

i.e there is an $a \in Q$ such that for all $b \in Q$

$a \leq b.$