Math 303, Fall 2011, Lecture 11

① The symbols of a formal language (Cohen I.2 (starting on p3))

Cohen talks of two kinds of symbols

general symbols common to all mathematics

and

symbols which are special to the kind of math we're doing

If we wish to use set theory as a foundation for all of mathematics then we only need special symbols for set theory. Every other special symbol will just an abbreviation for something which can be said using set theory, albeit perhaps in a very convoluted way.

So, our only special symbol is

$\in$ element of

And our general symbols are

$\sim$ (sometimes $\neg$) not

$\wedge$ (sometimes &) and

Note you can remember this is and because it looks like an A

$\vee$ or

$\rightarrow$ implies

Note elsewhere in math usually use double arrows $\Rightarrow$, $\Leftrightarrow$ in logic always single arrows

$\leftrightarrow$ if and only if

$\forall$ for all

$\exists$ there exists

(,) parentheses

$=$ equality

$x_0, x_1, x_2 \ldots$, $x, y, z, w, \ldots$ variable symbols

$c_0, c_1, \ldots$ $\ldots$ constant symbols

Notes

① Cohen also has names for relations but we don't need this because we can form relations as sets (just as for functions)

② We gave english intuitive meanings for each symbol but we'll give rules independent of the meanings to manipulate these. But keep the intuitive meanings in mind to have intuition of how to manipulate formally

③ If you want your language to be finite we need countably many variable symbols from finitely many actual symbols

solution: x, '

variables are $x, x', x'', x''', \ldots$

Same for constants $c, c', c'', c''', \ldots$

eg $\boxed{x \subseteq y}$ $\qquad$ $\boxed{\forall x' ((x' \in x) \to (x' \in y))}$

eg lets allow ourselves + in its usual sense for natural numbers (could define it as a function using sets)

$\boxed{x + y = y + x \text{ for all } x, y \in \omega}$

$$\forall x \forall y ((x \in \omega \wedge y \in \omega) \to (x + y = y + x))$$

here I'm using ω as a constant symbol

eg The emptyset

$$\exists x (\forall x' (\neg (x' \in x)))$$

One of our axioms tells us such an x exists so this statement is true

Knowing this - we can avoid having to say all this every time we need $\emptyset$

Just use ϕ knowing I could put in the whole statement

This is what is meant by an abbreviation

They aren't part of our formal system but they make things easier to read.

② The syntax of the formal system

Some things I can write with these symbols make sense

$$\forall z(z \in x \rightarrow z \in y)$$

Others do not

$$\forall \neg)\, x\, x\, x\, x\, z \in ((\rightarrow$$

But we want to be able to use these symbols without appealing to their meaning. So we need rules for forming valid strings

The valid strings are called well formed formulas (wff)

Here are the rules

1. For any x, y variable symbols and any c, d constant symbols. The following are well formed formulas

$$(x \in y)$$

$$(x \in c)$$

$$(c \in x)$$

$$(c \in d)$$

Note: These aren't necessarily true, just syntactically ok

eg $(x \in \emptyset)$

2. For x, y, c, d as above. The following are well formed formulas

$$(x = y) \qquad (c = d)$$

$$(x = c)$$

$$(c = x)$$

(3) If A and B are well formed formulas then so are

$(\neg A)$ $\quad (A \wedge B)$ $\quad (A \vee B)$ $\quad (A \rightarrow B)$
$(A \leftrightarrow B)$

(4) If A is a well formed formula and x is a variable symbol then

$(\forall x\, A)$ $\quad (\exists x\, A)$

Note again well formed formulas don't need to be true

eg $\quad (\forall x (\forall y (x = y)))$

is a well formed formula

But not true in set theory

to see this is well formed. (1) $(x = y)$ is well formed so (2) $(\forall y (x = y))$ is well formed so (3) $(\forall x (\forall y (x = y)))$ is well formed

Note We'll leave out parentheses when the meaning is clear

eg $\forall x \forall y (x=y)$ is clear

$\forall x \forall y\, x=y$ is not clear enough

eg Are the following well formed? (don't worry about parentheses if it's clear)

$x=y$ yes good

$x=y=z$ no $(x=y)\wedge(y=z)$ yes a wff

$\forall x(x\wedge y)$ $\forall x A$ where $\underbrace{A = x\wedge y}$, but $x\wedge y$ is not a well formed formula

$\forall x \exists y \exists z (y=z)$ yes not good x is not a free var in $\exists y \exists z (y=z)$

③ Next time

- free and bound variables, sentences
- examples
- truth

$\boxed{\text{Read Cohen Ch I sections 2 and 3}}$