Math 303, Fall 2011, Lecture 14

1. Predicate calculus

A week ago we had

Definition A propositional function on the letters $A_1, \dots, A_n$ is a string of symbols defined as follows:

1. A_i is a propositional function
2. If P and Q are propositional functions then so are

$$(\neg P), \; (P \wedge Q), \; (P \vee Q), \; (P \rightarrow Q), \; (P \leftrightarrow Q)$$

and two rules

Rule A Let P be a propositional function in the letters $A_1, A_2, \ldots, A_n$. If P is identically true then P with each A_i replaced by any sentence is a valid statement

Rule B If A and $A \to B$ are valid statements then so is B

But we need some more rules

a rule to encode the rules of equality

Rule C ① $c = c$, $(c = c') \to (c' = c)$, and

$$((c = c') \wedge (c' = c'')) \to (c = c'')$$

are valid statements for any three constant symbols c, c', and c''

② If A is a sentence, c and c' constant symbols and A' is A with every occurrence of c replaced by c', then

$$(c = c') \to (A \to A')$$

is a valid statement

a rule to encode change of variables

Rule D Let A be any sentence and x and x' variable symbols. Let A' be A with every occurrence of x replaced by x'. Then

$$A \leftrightarrow A'$$

is a valid statement

One consequence of Rule D is for any formula A we can find a good A' with $A \leftrightarrow A'$

And finally 3 rules about quantification

Rule E Let $A(x)$ be a formula in which every occurrence of the variable x is free

Let $A(c)$ be A with every occurrence of x replaced with the constant symbol c

Then

$$(\forall x\, A(x)) \to A(c)$$

is a valid statement for any constant symbol c

Rule F Let B be a sentence not involving the constant symbol c or the variable x. Then

if $A(c) \to B$ is valid

so is $\exists x\, A(x) \to B$

Rule G Let $A(x)$ have x as its only free variable and let every occurrence of x be free. Let B be a sentence which does not contain x. Then the following are valid statements

$$(\sim(\forall x\, A(x))) \leftrightarrow (\exists x(\sim A(x)))$$

$$((\forall x\, A(x)) \wedge B) \leftrightarrow (\forall x(A(x) \wedge B))$$

$$((\exists x\, A(x)) \wedge B) \leftrightarrow (\exists x(A(x) \wedge B))$$

eg let $A(x)$ be $x = c$

let B be $x = d$ $c \neq d$ constants

$$(\exists x\, A(x)) \wedge B \quad \not\leftrightarrow \quad \exists x(A(x) \wedge B)$$

not a sentence
no truth value

false

this shows why B must have no occurrences of x.

Definition Let S be a collection of statements. We say A is **derivable from S**, if for some $B_1, \ldots, B_n \in S$

$$(B_1 \wedge \ldots \wedge B_n) \to A \quad \text{is valid}$$

but this is not well formed
it is short for

$$B_1 \wedge (B_2 \wedge (\ldots (B_{n-1} \wedge B_n)) \ldots)$$

eg Say x does not appear in A. Show that A is derivable from $\forall x A$

ie $S = \{\forall x A\}$

this is asking if $\boxed{\forall x A \rightarrow A}$ is a valid statement

which it is by rule E

just sub any c in for x. Since A has no x then after subbing it hasn't changed

The next section in Cohen shows how valid statements correspond to true statements in some meaningful sense. But first lets go back to our axioms for set theory now that we have a formal language

② The axioms of set theory revisited

Axiom of extension two sets are equal if and only if they have the same elements

$$\forall x\, \forall y \left(\left(\forall z\, (z \in x \leftrightarrow z \in y) \right) \longleftrightarrow (x = y) \right)$$

Cohen ① Axiom of extensionality

Axiom of Specification or Subset Selection
For every set A and $\boxed{\text{every condition } S(z)}$ there is a set B consisting of exactly the elements of A for which $S(x)$ holds

ie $B = \{ z \in A \mid S(z) \text{ is true} \}$

for all formulas $\psi(z, t_1, \ldots, t_k)$ with at least one free variable (namely z)

extra parameters

$$\underbrace{\forall t_1 \ \forall t_k}_{\text{extra parameters}} \Big(\underset{A}{\forall x}\ \underset{B}{\exists y}\ \forall z \big(z \in y \leftrightarrow ((z \in x) \wedge \underset{S(z)}{\psi}(z, t_1, \ldots, t_k)) \big) \Big)$$

This is Cohen's 6′ (n) Axiom of separation found on p55

It is not just one axiom — we need one for each formula ψ. This is called an **axiom schema**

it is a family of axioms.

We don't have rules in our language to quantify over formulas

Axiom of the empty set There is a set, written $\emptyset$, which contains no elements

$$\exists y \, \forall x \, (\neg(x \in y))$$

This is Cohen's ② Axiom of the Null set

Axiom of pairing (or unordered pairs)

For any two sets A and B, there is a set C with $A \in C$ and $B \in C$ and nothing else

$$\underset{\substack{\uparrow \\ A}}{\forall x} \; \underset{\substack{\uparrow \\ B}}{\forall y} \; \underset{\substack{\uparrow \\ C}}{\exists z} \, \big((x \in z) \wedge (y \in z) \wedge (\forall w \, (w \in z \leftrightarrow (w = x) \vee (w = y))) \big)$$

or slightly shorter

$$\forall x \forall y \exists z \left(\forall w \left(w \in z \leftrightarrow (w = x) \vee (w = y) \right) \right)$$

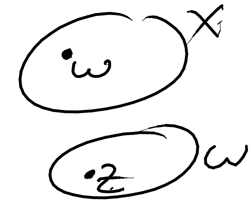
this is Cohen's ③ Axiom of unordered pairs

Axiom of Unions

Let $\mathcal{C}$ be a set of sets. Then there is a set which contains all elements which belong to at least one set from $\mathcal{C}$, and nothing else

$$\forall x \; \exists y \left(\forall z \; \exists w \left(z \in y \leftrightarrow (z \in w) \wedge (w \in x) \right) \right)$$

$\mathcal{C}$ $\quad$ $\bigcup \mathcal{C}$



This is Cohen's ④ Axiom of the sum set or union

Axiom of Power sets

For every set E, there is a set $\mathcal{P}$ consisting of precisely the subsets of E

that is $A \subseteq E$ if and only if $A \in \mathcal{P}$

$$\forall x \; \exists y \left(\forall z \left((z \subseteq x) \leftrightarrow (z \in y) \right) \right)$$

E $\quad$ $\mathcal{P}(E)$

this is an abbreviation we know.

This is Cohen's ⑦ Axiom of the Power set

Axiom of infinity

There exist a set containing (as an element) $\emptyset$ and containing (as an element) the successor of each of its elements

$$\exists x \left((\emptyset \in x) \wedge \forall y \left((y \in x) \rightarrow (\underbrace{y \cup \{y\}} \in x) \right) \right)$$

some successor set

$\emptyset$ is an abbreviation we defined it in a previous axiom

lets check we can say this

$z = y \cup \{y\}$ abbreviates

$$\forall w \left((w \in z) \leftrightarrow \left((w \in y) \vee (w = y) \right) \right)$$

we're checking we can say y^+

This is Cohen's

5 Axiom of infinity

And finally we have the axiom of choice

The axiom of choice

Let I be a nonempty set. Let $\{Y_i\}_{i\in I}$ be a family of nonempty sets indexed by I.
Then
$$\bigtimes_{i\in I} Y_i \neq \emptyset$$

Cohen gives it in terms of a choice function ⑧ Axiom of Choice

To look more like Halmos we could write

have this abbreviation

$$\forall I \, \forall f \, \Big(\underbrace{f \text{ is a family indexed by } I}_{\exists X (f \in X^I)} \wedge (\neg I = \emptyset) \wedge \forall i \big(i \in I \to (\neg \underbrace{f(i)} = \emptyset)\big)\Big)$$

$$\to \neg \Big(\underbrace{\bigtimes_{i\in I} f(i)} = \emptyset \Big)\Big)$$

$\exists X (f \in X^I)$

we could expand this ordered pairs give functions

again the cartesian product was a particular set of families

once f is written as a set $f(i)$ is the x such that $x \in X \wedge (i,x) \in f$

Cohen has 2 more axioms – the remaining two which make Zermelo set theory into Zermelo-Fraenkel set theory

(9) Axiom of regularity

$$\forall x \, \exists y \, (x = \emptyset \vee ((y \in x) \wedge \forall z (z \in x \to \sim (z \in y))))$$

ie y is minimal in x with respect to $\in$

$(6)_n$ Axiom of replacement

For every formula $\psi(x, y, t_1, \dots, t_n)$ with at least 2 free variables (x and y) we have

$$\forall t_1 \forall t_2 \dots \forall t_k \Big(\overbrace{\forall x \, \exists! y \, (\psi(x, y, t_1, \dots, t_k))}^{\psi \text{ is acting like a function } x \mapsto y} \to \forall \underbrace{u}_{\text{input set}} \exists \underbrace{v}_{\text{output really is a set}} B(u, v) \Big)$$

where $\exists! y$ is an abbreviation for

there exists a unique y

ie $\exists y \big(\psi(x, y, t_1, \ldots, t_k) \to \forall u \exists v \, B(u,v)$

$\wedge \forall z \big(\psi(x, z, t_1, \ldots, t_k) \to y = z\big)\big)$

$f(x) = y$

$f(x) = y'$

if f is a function

need $y = y'$

and $B(u,v)$ is an abbreviation for

$$\forall r \big(r \in v \longleftrightarrow \exists s \big((s \in u) \wedge \psi(s, r, t_1, \ldots, t_k)\big)\big)$$

what this means is that the images of functions applied to sets are sets

ie if X is a set, f a function

then $\{ y : \exists x \, (f(x) = y) \}$ is a set.

So in plain english we could say ranges of functions are sets

We'll talk about this more when we get to Halmos' version in Halmos section 19.

Note this is also an axiom schema. Also the axiom of subset selection is a consequence

To see this (sketch) choose a set c

define a function

$$f(x) = \begin{cases} x & \text{if } x \text{ satisfies the property } S(x) \\ c & \text{otherwise} \end{cases}$$

The axiom of replacement says the image of A under f is a set. But this image is

$$\{x \in A \mid S(x) \text{ true}\} \cup \{c\}$$

So if we want $\{x \in A \mid S(x) \text{ is true}\}$ ie subset selection

just choose $c \notin A$

then $\underbrace{(\text{image of } A \text{ under } f)}_{\text{is a set by axiom of replacement}} - \overbrace{\{c\}}^{\text{is a set by pairing}}$

$= \underbrace{\{x \in A \mid S(x) \text{ true}\}}$

and so this is a set.