Math 303, Fall 2011, Lecture 18

1 Important results of Model theory

Here are some important results of model theory. We won't prove them (see Cohen if you are interested).

Definition A set of statements is **consistent** if the statement $A \boxed{\wedge} \sim A$ cannot be derived from S for any A

1. If A is a valid statement then A is true in any model

2. If a set of statements has a model then it is consistent

not to be confused with the incompleteness theorem

③ (Gödel's Completeness theorem)

Let S be any consistent set of statements. There exists a model M for S for which

$$\#(M) \leq \begin{cases} \#(S) & \text{if } \#(S) \geq \#(\omega) \\ \boxed{\#(\omega)} & \text{if } \#(S) < \#(\omega) \end{cases}$$

this # is cardinality we'll get to that in a week and a half or so.

For now think of # as size where the size of any countably infinite set is the size of ω

$\#\omega$

④ Let S be any set of statements. If A is not derivable from S then there is a model for S in which A is false

⑤ (Compactness theorem) Let S be a set of statements. If every finite subset of S has a model, then S has a model

Definition Let $M_1 \leq M_2$ be models of S where the interpretations of the constants are the same and for each relation R

this says the interpretations are the same on both sets.

$$(\overline{R} \text{ in } M_2) \cap M_1 = \overline{R} \text{ in } M_1$$

If for every formula $A(x_1, \dots, x_n)$ and every $\overline{x}_1, \dots, \overline{x}_n$ in M_1

$$\begin{matrix} A \text{ is true in } M_1 \\ \text{at } \overline{x}_1, \dots, \overline{x}_n \end{matrix} \iff \begin{matrix} A \text{ is true in } M_2 \\ \text{at } \overline{x}_1, \dots, \overline{x}_n \end{matrix}$$

Then M_1 is an $\boxed{\text{elementary}}$ submodel of M_2

Idea a elementary submodel is one which is the same in all its logical aspects.

⑥ (Löwenheim - Skolem) Let T be a set of constant symbols and relation symbols.

Let M be a model of these symbols (ie $S=\emptyset$ but have an interpretation for the symbols of T)

Then there is an $\boxed{\text{elementary}}$ submodel N of M with

$$\#(N) \le \begin{cases} \#(T) & \text{if } \#(T) \ge \#(\omega) \\ \#(\omega) & \text{if } \#(T) < \#(\omega) \end{cases}$$

Note since N is an elementary submodel, all sentences true in M are true in N and vise versa. So don't need an S

This is all we're going to say about logic for now (come back with Skolem's paradox)

Next - back to Halmos

② Review of partial orders
(Halmos chapter 14)

Recall that if X is a set and $\leq$ is a relation on X and the following properties are satisfied

for all a, b, c in X

① if $a \leq b$ and $b \leq a$ then $a = b$

② if $a \leq b$ and $b \leq c$ then $a \leq c$

③ $a \leq a$

then $\leq$ is called a partial order and X is called a partially ordered set (or poset)

We've seen 3 examples of this

⓪ Usual $\leq$ on ω (or any other total order)

① divisibility : $\{1, 2, \dots\}$ is partially ordered by divisibility

(see lecture 10)

② Let E be any set, $P(E)$ is partially ordered by $\subseteq$
this was P2 on assignment 4
(see solutions if you didn't do it on the assignment)

These are the two most important examples and the ones you should base your intuition on.

Definition Let X with $\leq$ be a partially ordered set

① $a \in X$ is a least or smallest element of X if for all $x \in X$, $a \leq x$

② $a \in X$ is a greatest or largest element of X if for all $x \in X$ $x \leq a$

③ $a \in X$ is a minimal element of X if there is no element of X which is strictly smaller than a

equivalently if $x \in X$ with $x \leq a$ then $x = a$

④ $a \in X$ is a maximal element of X if for all $x \in X$, if $x \geq a$ then $x = a$

eg Take $X = \{2, 3, 4, \dots\}$ partially ordered by divisibility

least elements: no least element $2 \nmid 3$ so 2 is not least. And nothing else can be least as a number can't divide a number which is smaller than it

minimal elements: 2 (nothing in the set divides 2)
in fact all the primes are minimal in X

eg let E be a set with at least 2 elements
let $X = P(E) - E$ (so X is the set of proper subsets of E)
partially order X by $\subseteq$
Then X has maximal elements but no greatest element

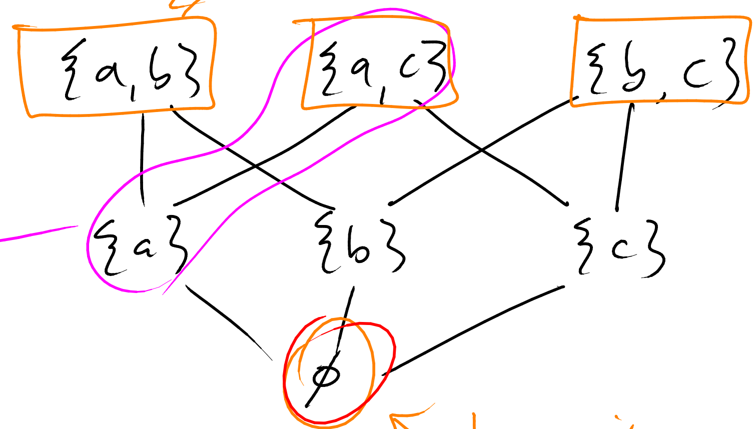
sub eg let $E = \{a, b, c\}$
then $X = \{\emptyset, \{a\}, \{b\}, \{c\}, \boxed{\{a,b\}}, \{a,c\}, \{b,c\}\}$
maximal elements $\{a,b\}$, $\{a,c\}$, $\{b,c\}$
nothing is bigger than them.

here is a picture as

no greatest element
nothing is bigger than
everything else.

here are the maximal elements; nothing is above them

eg this line shows $\{a\} \subseteq \{a,c\}$

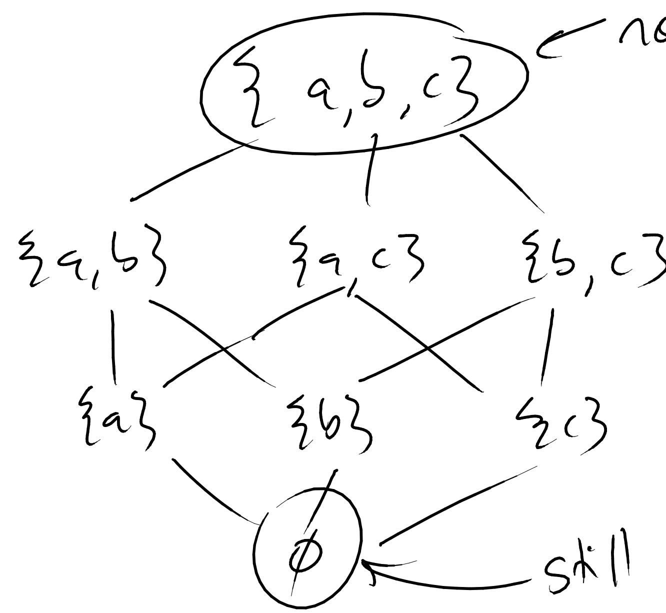


bigger in $\subseteq$.

here is a minimal element (nothing is below it)

it is also least (it is below everything)

In contrast if $X = \mathcal{P}(E)$ then the picture is



now this is both greatest and maximal.

still least and minimal.

③ Next time

- Review of well orders
- Transfinite induction
- Ordinals

Please read Halmos sections 17 and 19