Math 303, Fall 2011, Lecture 19

① Review of well orders

Recall that if X with $\leq$ is a partially ordered set and

④ for all $x, y \in X$, $x \leq y$ or $y \leq x$
⑤ every subset $Y \subseteq X$ has a least element

then we say $\leq$ is a well order and we say X is well ordered by $\leq$

Note ④ isn't necessary to define a well order as take $\{x,y\} \subseteq X$

$\{x,y\}$ must have a least element

if x is the least element then $x \leq y$
if y is the least element then $y \leq x$

Note If a partial order satisfies (4) (but not necessarily (5)) we say it is a total order

We saw that ω is well ordered by the usual $\leq$

eg Consider $\omega \times \omega$

What ordering shall we use?

(a) say $(a,b) \leq (c,d)$ if $a \leq c$ and $b \leq d$

eg $(1,3) \leq (2,867)$

$(1,3) \not\leq (4,2)$

Is $\omega \times \omega$ well ordered?

Answer no eg $\{(1,3),(4,2)\}$ has no least element

(b) Say $(a,b) \leq (c,d)$ if $a < c$

or

$a = c$ and $b \leq d$

this is called lexicographic order because it is how we put things in alphabetical order

Is $\omega \times \omega$ well ordered?

yes let $X \subseteq \omega \times \omega$

let $Y \subseteq X$ be the pairs $(a,b) \in X$ such that for all $(c,d) \in X$

$a \leq c$

(no restriction on c)

$Y \neq \emptyset$ since ω is well ordered

but then the elements of Y differ only in the second coordinate, so forget the first coordinate and since ω is well ordered we get a least element of Y

which is here also a least element of X

(2) Transfinite induction

Well ordered sets have the nice property that we have a notion of induction. First we need the following definition

Definition Let X with $\leq$ be a partially ordered set. Take $a \in S$. The set

$$S(a) = \{x \in X : x < a\}$$

is called the initial segment of a

Let X be a well ordered set and let $S \subseteq X$
if for all $x \in X$

$$s(x) \subseteq S \quad \text{implies} \quad x \in S$$

Then $S = X$

This is called the principle of transfinite induction

First lets see that this is true and then see what we can do with it

To check the fact suppose S has the property but $S \neq X$. Then $X \setminus S$ is not empty. But X is well ordered so $X \setminus S$ has a least element x. Since x is least in $X \setminus S$ $s(x) \subseteq S$. But then $x \in S$ contradiction.

How does this relate to the principle of mathematical induction which we have already seen?

let $X = \omega$

First note that transfinite induction checks the base case automatically

Take $0 \in \omega$

$s(0) = \emptyset$ so $s(0) \subseteq S$. Thus $\boxed{0 \in S}$

we don't need a separate statement to check the base case — it is built in

Next note that for $X = \omega$ transfinite induction is strong induction, we must assume the whole initial segment of x to conclude the result for x.

The usual principle of mathematical induction is weak induction, we just assume $n-1$ to get n.

For ω strong and weak induction have the same power (so it is silly to make the distinction)

But for other well ordered sets transfinite induction is necessary

eg let $X = \omega^+ = \omega \cup \{\omega\}$

use the usual ordering on ω along with $n < \omega$ for $n \in \omega$

This is a well ordering

Suppose we try to use the old principle of mathematical induction on X. What goes wrong?

For the break

$\boxed{\text{Can you find an } S \subsetneq \omega^+ \text{ with } 0 \in S \text{ and for all } n \in S,\ n^+ \in S\text{?}}$

answer

Transfinite induction fixes this problem

$$s(\omega) = \omega \subseteq S$$

so we must have $\omega \in S$, hence $S = X$

③ Ordinals

We had

0

$0^+ = 1$

$1^+ = 2$

$2^+ = 3$

$\vdots$

all together this is $\omega = \{0, 1, \dots\}$

Now consider

$$\omega, \quad \omega^+, \quad (\omega^+)^+, \quad ((\omega^+)^+)^+$$

where does this go?
is there something beyond all this the same way ω is beyond all natural numbers?

Suppose f is a function with domain $n \in \omega$

Say f is an ω-successor function
if $f(0) = \omega$
and $f(m^+) = (f(m))^+$

eg $n = 3 = \{0, 1, 2\}$.

$$f(0) = \omega$$
$$f(1) = f(0^+) = (f(0))^+ = \omega^+$$
$$f(2) = f(1^+) = (f(1))^+ = (\omega^+)^+$$

In fact for each n there is a unique ω-successor function

Suppose f and g were both ω-successor functions with domain n

- $f(0) = g(0) = \omega$
- Let $i \in n$ be the smallest number for which $f(i) \neq g(i)$.
- $i > 0$ so $i = j^+$ for some $j \in n$

  but $f(j) = g(j)$ since i was minimal

  so $f(i) = f(j^+) = (f(j))^+ = (g(j))^+ = g(j^+) = g(i)$

  contradiction

Thus f is unique.

What we want is to join all those things together

Let $S(n,x)$ be the property

"$n\in\omega$ and x is in the range of an ω-successor function with domain n"

in logic

$$n\in\omega \wedge \exists f\Big(\big(f \text{ is an } \omega\text{-successor function with domain } n\big)\wedge \exists a\big((a,x)\in f\big)\Big)$$

where "f is an ω-successor function with domain n" can be written

This is short for
$\forall z\forall t\forall u$
$\big((z\in f \wedge z=(t,u)) \leftrightarrow \text{rest}\big)$

$$\forall (t,u)\in f\,\underbrace{\Big[(t=0\wedge u=\omega)\vee\big((t\in n)\wedge \exists y\exists w(y^+=t \wedge (y,w)\in f \wedge w^+=u)\big)\Big]}_{\omega\text{-successor property}}$$

$$\wedge\ \underbrace{\forall v\big((f,v)\in f \leftrightarrow v=u}_{\text{function property}}\big)$$

The set we are looking for is

$$\{x : \exists n \in \omega\, (S(n,x))\}$$

We only know this is a set by using the axiom of replacement

Intuitively S is acting like a function F with domain ω defined by

$$F(n) = \{x : S(n,x)\}$$

We want to know

either (a) F actually is a function in the set theoretic sense

or (b) The image of any $X \subseteq \omega$ under F is a set

These are equivalent. If F is a set theoretic function then its range is a set and so by subset selection so is the image of any $X \subseteq \omega$

On the other hand if all the images are sets, then its range Y is a set and so $\mathcal{P}(\omega \times Y)$ is a set and so we can pull out f by subset selection

(b) would come from Cohen's version of the axiom of replacement

(a) is Halmos' version which he calls the axiom of substitution

Axiom of Substitution If $S(a,b)$ is a sentence such that for each $a \in A$ the set $\{b : S(a,b)\}$ can be formed, then there exists a function F with domain A such that $F(a) = \{b : S(a,b)\}$ for each $a \in A$

Our use of the axiom of substitution will be, as above to extend our ability to count beyond $\omega, \omega^+, \ldots$

So we have

$$0, 1, 2, 3, \ldots\ldots$$

$$\omega, \omega^+, (\omega^+)^+, ((\omega^+)^+)^+, \ldots \qquad (*)$$

and by the above we can define a set theoretic function F with domain ω such that

$$F(0) = \omega, \quad F(n^+) = (F(n))^+$$

Let X be the range of F

Then the next number after the ones in $(*)$ is

$$X \cup \omega = \{0, 1, 2, \ldots, \omega, \omega^+, (\omega^+)^+, \ldots\}$$

We want to all that work with the axiom of replacement just to check that $X \cup \omega$ really is a set

What are these new bigger counting "numbers"

ordinals

Definition An ordinal is a well ordered set S such that for all $x \in S$ $\quad s(x) = x$

eg lets check 3 is an ordinal

$$3 = \{0, 1, 2\}$$

use the usual order $\quad 0 < 1 < 2$

This is a well order (check)

Now check the ordinal property

$0: \quad s(0) = \emptyset, \quad 0 = \emptyset$

$1: \quad s(1) = \{0\}, \quad 1 = \{0\}$

$2: \quad s(2) = \{0, 1\}, \quad 2 = \{0, 1\}$ $\quad$ so it works

Likewise every natural number is an ordinal.

eg check ω is an ordinal.

Again use the usual order which we already noticed is a well order

Take $n \in \omega$

then $s(n) = \{0, .., n-1\} = n$

So ω is an ordinal.

Two useful facts

① If X is an ordinal then X^+ is an ordinal

proof Use the order on X^+ given by

for $x, y \in X$, $x \leq y$ in X^+ if and only if $x \leq y$ in X

for $x \in X$, $x \leq X$ in X^+

This is a well order as if $Y \subseteq X^+$

then if $X \in Y$, Y has a least element since X was well ordered

and if $X \in Y$ then either $Y = \{X\}$ with least element X

or $Y = Y' \cup \{X\}$ where $Y' \subseteq X$ has a least element which is hence a least element of Y.

Finally we can check the ordinal property.

Take $x \in X^+$. If $x \in X$ we already know $s(x) = x$

If $x = X$ then $s(x) = X$

② Let X be a set. There is at most one well order which makes X into an ordinal

proof Suppose there is a well order which makes X into an ordinal. Take any other well order of X

at least one $x \in X$ has a different initial segment in the new order

but $s(x) = x$ in the old order

so $s(x) \neq x$ in the new order

Thus the new order does not make X into an ordinal.

④ Next time

More on ordinals

please read Halmos section 20