Math 303, Fall 2011, Lecture 21

① Ordering the ordinals

Last time we saw that two well ordered sets X and Y are similar if there is a function $f: X \to Y$ which is

$\vdots$

and

Let X and Y be well ordered sets. Either X and Y are similar or one of them is similar to an initial segment of the other

And not both

For ordinals we get one further similarity property

If two ordinals are similer Then they are equal

proof let X and Y be ordinals and

$f : X \to Y$ a similarity

let $S = \{ x \in X : f(x) = x \}$

Use transfinite induction

so we need to check

Putting all this together we get

$\boxed{\text{If } X \text{ and } Y \text{ are ordinals then either } X = Y \text{ or one of them is equal to an initial segment of the other}}$

This uses

Thus

$$X < Y$$

In fact $\boxed{\text{this ordering is a well ordering}}$

proof let E be any nonempty set of ordinals.
Take any $X \in E$.

Note that $Y \in X \cap E$ so in particular $X \cap E \neq \emptyset$

Furthermore the order on ordinals satisfies the ordinal property!!

Question Is the set of all ordinals an ordinal?

Think about it.

This is called the Burali-Forti paradox

The Burali-Forti paradox is similar to Russell's paradox but predates it (by a few years)

1897 1901

though all these ideas were in the air at the same time.

(2) Counting with ordinals

We have

write write

Next is

← this notation comes from

then

← note

And after you run out of "arithmetic" notation the next one is called

ε_0 and on it goes $\varepsilon_0 + 1, \ldots$

For the break

is ε_0 countable ?

② Size

A while ago we said two sets were the same size if

The ordinals don't capture this notion of size

eg

Similarly

Also $\omega 2$ is the same size as ω

$\{0, 1, 2, 3, \dots, \omega, \omega+1, \omega+2, \dots\}$

$\{0, 1, 2, 3, 4, 5, 6 \dots\}$

And so on

Two natural questions

1.

2.

Recall that we say two sets are equivalent if they are the same size in this sense

We will write $X \sim Y$ to say X is equivalent to Y

(compare to similarity for well ordered sets)

Let X and Y be sets. Say Y dominates X and write

$$X \preccurlyeq Y$$

a curly $\leq$

if X is equivalent to a subset of Y

We would like

•

•

This is sufficiently important to be a named theorem

Schröder - Bernstein Theorem

If $X \precsim Y$ and $Y \precsim X$ then $X \sim Y$

This tells us

The theorem says

I'm not going to prove Schröder-Bernstein because I want to have time to get to some more paradoxes

Another useful fact is

if X and Y are sets then either $X \precsim Y$ or $Y \precsim X$

proof well order X and Y

either

or

③ Countability revisited

Write $X \prec Y$ to mean $X \precsim Y$ and $X \not\approx Y$

With this notion of domination, we can say

X is finite if

X is infinite if

X is countable if

X is countably infinite if

Note

Some things which are countable

① The union of two countable sets is countable

$$X \xrightarrow{f} \omega \xleftarrow{g} Y \qquad f, g \text{ one-to-one}$$

$$\omega = \{0, 1, 2, \dots\}$$

(2) The cartesian product of two countable sets is countable

$$X \xrightarrow{f} \omega \xleftarrow{g} Y$$

$$\omega = \{0, 1, 2, \ldots\}$$

f, g one-to-one

(3) The set of finite subsets of a countable set is countable

$$\begin{array}{c} X \\ f \big\downarrow \\ \omega = \{0, 1, 2, \dots\} \end{array}$$

f one-to-one

④ Next time

- Cantor diagonalization – not every set is countable
- Cardinal numbers

$\boxed{\text{Please read Halmos section 25}}$