Math 303, Fall 2011, Lecture 23

① Cardinal numbers

We want a distinguished set of each size which we'll use to represent that size. This is the same idea as how we chose a particular set of size n to represent the natural number n.

We have many ordinals of each size. But the ordinals are well ordered. So pick the smallest ordinal of a given size to be the cardinal of that size.

Definition A **cardinal number** is an ordinal number X such that if Y is an ordinal number and $X \sim Y$, then $X \leq Y$

ordinal $\leq$ not domination $\preceq$

same size equivalent (not similar)
(ie forget order)

If Z is any set, then $\#Z$, the cardinality of Z is the least ordinal equivalent to Z

this is what we've been casually calling size

eg What is $\#3$? (it had better be 3 or this theory is really stupid)

3 is itself an ordinal

further there are no other ordinals of size 3

so 3 is the smallest ordinal equivalent to 3

so $\#3 = 3$

The same holds for every natural number

eg What is $\#\mathbb{Z}$? (set of integers)

we know $\omega \sim \mathbb{Z}$ and ω is an ordinal

Furthermore ω is the smallest infinite ordinal so no ordinal of the same size is smaller so ω is also a cardinal

$$\#\mathbb{Z} = \omega$$

When we view ω as a cardinal we usually give it a different name

$\aleph_0$ this is the Hebrew letter aleph with a subscript 0

$\aleph$

Some facts about cardinals

① let X and Y be sets

$$\#X = \#Y \iff X \sim Y$$

Note this fact says Cardinality really does measure size

proof Any set is equivalent to its cardinal number

so $X \sim \#X$ and $Y \sim \#Y$

thus if $\#X = \#Y$ then $X \sim \#X = \#Y \sim Y$

so $X \sim Y$

while if $X \sim Y$ then $\#X \sim X \sim Y \sim \#Y$ so $\#X \sim \#Y$

so both $\#X$ and $\#Y$ are ordinals which are smallest among equivalent ordinals. But the smallest element is unique so $\#X = \#Y$.

2 There is an ordering on cardinals which makes any set of cardinals well ordered

proof Cardinals are special ordinals and any set of ordinals has this property so the cardinals have this property

② The continuum hypothesis

How do we count with cardinals?

$$0, 1, 2, 3, \ldots, \aleph_0$$

$$\aleph_0 = \omega \sim \omega+1 \sim \omega+2 \sim \ldots \sim \omega 2$$

$$\sim \omega 3 \sim \ldots \sim \omega^2 \ldots \sim \omega^\omega \sim \ldots \sim \varepsilon_0$$

Let $\aleph_1$ be the next cardinal (the first uncountable cardinal)

We also know $2^{\aleph_0}$ is uncountable

$$2^{\aleph_0} = \# 2^{\mathbb{Z}} = \# \mathbb{R}$$

Certainly $\aleph_1 \leq 2^{\aleph_0}$ (since $\aleph_1$ is smallest uncountable cardinal)

The continuum hypothesis says $\aleph_1 = 2^{\aleph_0}$

If not then there is an uncountable set smaller (in cardinality) than the real numbers.

Is the continuum hypothesis true or false?

Why the word continuum?

the real line is called the continuum

and $2^{\aleph_0} = \# \mathbb{R}$

Georg Cantor formulated the continuum hypothesis in 1874

In 1900 determining whether the continuum hypothesis was true or false was the first of Hilbert's problems

The resolution though was far from what Hilbert would have imagined

In 1940 Kurt Gödel showed that the continuum hypothesis is consistent with the axioms of Zermelo-Fraenkel set theory

In 1963 Paul Cohen showed that the negation of the continuum hypothesis is also consistent with the axioms of Zermelo-Fraenkel set theory.

Thus

THE CONTINUUM HYPOTHESIS IS

INDEPENDENT

OF THE AXIOMS OF SET THEORY

I think this is one of the craziest results of modern mathematics

Did you notice that the Halmos book is too old to know this?

It is about a semester of work to go through the proof and another to build some model theoretic background!

(One source of the proof is the last third or so of Cohen's book)

lets have a break now

③ Skolem's paradox

from lecture 18 we had

⑥ (Löwenheim - Skolem) Let T be a set of constant symbols and relation symbols.

Let M be a model of these symbols

(ie $S = \phi$ but have an interpretation of these symbols)

Then there is an elementary submodel N of M with

$$\#(N) \leq \begin{cases} \#(T) & \text{if } \#(T) \geq \#(\omega) \\ \#(\omega) & \text{if } \#(T) < \#(\omega) \end{cases}$$

where an elementary submodel is one in which the same sentences are true

Now lets apply this to set theory

In set theory we have one relation symbol $\in$

So by Löwenheim-Skolem we have a countable model of set theory with the same true sentences as in usual set theory

But from Cantor diagonalization we know there are uncountable sets so

"there is an uncountable set"

is true in usual set theory

Thus "there is an uncountable set" is true in the countable model of set theory

this is Skolem's Paradox

How do we resolve Skolens paradox?

Saying X is uncountable is saying there is no one-to-one function $f: X \to \omega$

But functions are also sets.

In the countable model we have fewer sets and in particular the function we need isn't there

So X can be uncountable in the countable model

But in the larger set theory we have the extra functions we need and so we see X is actually countable viewed in usual set theory

Skolem pointed out this paradox in 1922

Skolem felt that this result showed that first order logic was not the right tool for the foundations of mathematics.

So far history has disagreed, but the picture is certainly not as clean as people hoped at the time

But the lack of cleanness – the paradoxes – are beautiful in their own right hence this course.

And that's the end. Thanks for coming.
Monday will be review for the final.