Questions before we begin

When should the midterm be?

Oct 13	17
Oct 20	16
Oct 27	18
~~Nov 3~~	23
Nov 10	10

will finalize this in 1 week

What about a discussion list / web forum?

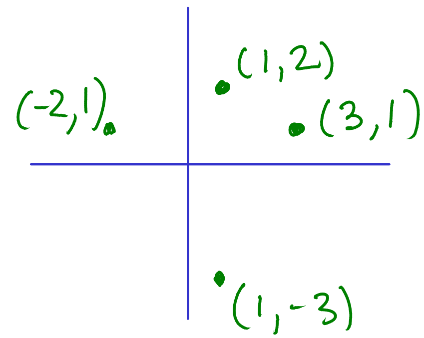
	pro	con
webct	easy to set up familiar consistent with other courses	crummy
google groups		
our own web thing phpbb	anymous possible	work to setup time to be done

piggy back on
physics forums
etc.

Math 303, Fall 2011, Lecture 4

① Ordered pairs

think of x, y coordinates (1, 5) eg



What do we need our ordered pair to do?
what properties should it have

ordered need $(a,b) \neq (b,a)$ ✓ for $a \neq b$

2 pieces of information
1 st coord
2 nd coord ✓

what kind of things go into the first and 2^{nd} coords?
- Answer sets because everything in this course is a set

need $\boxed{(a,b) = (c,d)}$
then $a = c$, $b = d$

The standard answer is $(a,b) = \boxed{\{\{a\}, \{a,b\}\}}$

Note this is not the right answer. Many answers are possible.
As long as it has the right properties it will do.

Halmos says:

It is easy to locate the source of the mistrust and suspicion that many mathematicians feel toward the explicit definition of ordered pair given above. The trouble is not that there is anything wrong or anything missing; the relevant properties of the concept we have defined are all correct (that is, in accord with the demands of intuition) and all the correct properties are present. The trouble is that the concept has some irrelevant properties that are accidental and distracting. The theorem that $(a, b) = (x, y)$ if and only if $a = x$ and $b = y$ is the sort of thing we expect to learn about ordered pairs. The fact that $\{a, b\} \in (a, b)$, on the other hand, seems accidental; it is a freak property of the definition rather than an intrinsic property of the concept.

This is very typical of the things we will build in set theory

Now lets check our definition really does capture what it should mean to be an ordered pair.

That is, we want that if $(a,b) = (x,y)$

then $a = x$ and $b = y$

So suppose $(a,b) = (x,y)$

$$\underbrace{\{\{a\},\{a,b\}\}}_{} \qquad \{\{x\},\{x,y\}\}$$

if (a,b) has one element

then $\{a\} = \{a,b\}$ this implies $a = b$

so $(a,b) = \{\{a\}\}$

Also if $(a,b) = (x,y)$ then (x,y) has just one element

so by the same reasoning $(x,y) = \{\{x\}\}$, $x = y$

So $\{\{a\}\} = \{\{x\}\}$ but a set is determined by its elements so $\{a\} = \{x\}$ so $a = x$

so $a = b = x = y$

if (a,b) has 2 elements

then $a \neq b$

$$\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$$

and a set is determined by its elements so

$$\underbrace{\begin{aligned}\{a\} &= \{x\} \\ \{a,b\} &= \{x,y\}\end{aligned}}$$

or

$$\underbrace{\begin{aligned}\{a\} &= \{x,y\} \\ \{x\} &= \{a,b\}\end{aligned}}_{\text{impossible as}}$$

$a \neq b$ so $\{x\}$ has one element but $\{a,b\}$ has 2 elements

so $a = x$ (top)

and so $b = y$ (bottom and)

How do we pull out the first and second coordinates?

If (a,b) has one element then $(a,b) = \{\{\boxed{a}\}\}$ (see above)

so $a=b$ and we can find a here

If (a,b) has 2 elements then

the first coordinate is

- the unique element contained in both elements of (a,b)

or

- the element of the singleton

then the 2nd coord is the other one

eg $(1,2) = \{\{1\},\{1,2\}\}$

eg $(\{\emptyset\},\emptyset) = \{\{\{\emptyset\}\},\{\{\emptyset\},\emptyset\}\}$

eg what ordered pair is $\{\{c,18\},\{18\}\}$? $(18,c)$

eg what ordered pair is $\{\{1,\{2,\{3\}\}\},\{\{2,\{3\}\}\}\}$

$= (\{2,\{3\}\},1)$

How to we build ordered pairs just with our axioms so far.

given a, b

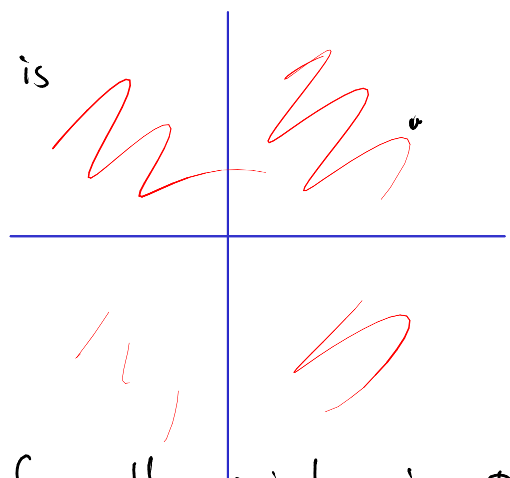
build $\{a\}$ by pairing a with a $\{a,a\} = \{a\}$

build $\{a,b\}$ " a with b $\{a,b\}$

build $\{\{a\},\{a,b\}\}$ " $\{a\}$ with $\{a,b\}$ get answer.

② Cartesian products

The cartesian plane is



our good old coordinate plane.

It is the set of all points in the plane

ie the set of ordered pairs of real numbers

In general If X and Y are sets do all ordered pairs (x,y) with $x \in X$ and $y \in Y$ form a set

Yes

Suppose $a \in A$ and $b \in B$

$\{a\} \subseteq A \subseteq A \cup B$ so $\{a\} \in \mathcal{P}(A \cup B)$

$\{a, b\} \subseteq A \cup B$ so $\{a,b\} \in \mathcal{P}(A \cup B)$

So $(a,b) = \{\{a\}, \{a,b\}\} \subseteq \mathcal{P}(A \cup B)$

so $(a,b) \in \mathcal{P}(\mathcal{P}(A \cup B))$

So all orderd pairs are elements of $\mathcal{P}(\mathcal{P}(A \cup B))$

so all orderd pairs form a set, a subset of ↑

Call it $A \times B = \{ p \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid p = (a,b)$ for some $a \in A$, $b \in B \}$

the cartesian product of A and B

Now use specification to cut it down to just ordered pairs

Define the **cartesian product** of A and B to be

$$A \times B = \{ x \in \mathcal{P}(\mathcal{P}(A \cup B)) : x = (a,b) \text{ for some } a \in A \text{ and some } b \in B \}$$

eg if $\mathbb{R}$ is the set of real numbers then $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the usual cartesian plane

eg $A = \{1\}$, $B = \{2,3\}$

$$A \times B = \{(1,2),(1,3)\} = \{\{\{1\},\{1,2\}\},\{\{1\},\{1,3\}\}\}$$

$\in \mathcal{P}(A \cup B)$ $\in \mathcal{P}(A \cup B)$

$\subseteq \mathcal{P}(A \cup B)$

$\in \mathcal{P}(\mathcal{P}(A \cup B))$

$\subseteq \mathcal{P}(\mathcal{P}(A \cup B))$

$$B \times A = \{(2,1),(3,1)\}$$

eg $A = \emptyset$ $B = \{1,2,3\}$

no elements can go here, not even $\emptyset$

$$(\;\; , 1)$$

$$A \times B = \emptyset$$

③ Next time

A similarly artificial, but does what we need definition of numbers

Please read Halmos sections 11, 12, and 13