Math 303, Fall 2011, Lecture 5

① Numbers

We want to build nonnegative integers out of sets.
As with ordered pairs our construction must have the properties we want but may also have spurious properties

Idea want 5 to be a particular set of 5 elements
that was easy to make using 4, 3, 2 ...

Start with 0   want 0 to be a set with 0 elements
there's only one such set.

Define $0 = \emptyset$

How about 1 ?

We need a set with 1 element

There are lots of those

$\{\phi\}$, $\{\{\phi\}\}$, $\{\{\phi, \{\phi\}\}\}$

etc

but we want the simplest one we can make out of 0. So

$$\boxed{\text{Define} \quad 1 = \{\phi\}}$$

Notice $1 = \{0\}$

What about 2

$2 = \{\{\phi\}, \phi\}$

What about 2?

Define $2 = \{\phi, \{\phi\}\}$
$= \{0, 1\}$

What about 3?

Define $3 = \{0, 1, 2\} = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$

How do we write this in general?

given $0, 1, \dots, n-1$ define

$$n = \{0, 1, 2, \dots, n-2, n-1\} = \{ i \in \mathbb{N} \mid 0 \le i \le n-1 \}$$

note $3 \subseteq n$ for $n \ge 3$

also $n-1 \subseteq n$

so $\boxed{n = (n-1) \cup \{n-1\}}$

One way to phrase it:

not necessarily a natural number

Definition For every set x, define the successor of x to be

$$x^+ = x \cup \{x\}$$

eg $2^+ = \{0,1\} \cup \{2\} = \{0,1,2\} = 3$

eg $\{\{\emptyset\}\}^+ = \{\{\emptyset\}\} \cup \{\{\{\emptyset\}\}\} = \{\{\emptyset\}, \{\{\emptyset\}\}\}$

Note not necessarily a number

eg $n = (n-1) \cup \{n-1\}$

try $n = 3$

$$3 = 2 \cup \{2\}$$
$$= \{0,1\} \cup \{2\}$$
$$= \{0,1,2\}$$

We call $0, 1, 2, \ldots$ natural numbers

Note: Some conventions start the natural numbers at 1. In set theory we always want to start with 0

Some weird properties of natural numbers

(1) Every element of a natural number n is also a subset of n

eg $n = 5 = \{\boxed{0, 1, 2}, 3, 4\}$

$\{0,1,2\} = 3 \subseteq 5$

$3 \in 5$

(2) If m and n are natural numbers then

$m \leq n$ or $n \leq m$ (both if $m = n$)

Some good properties of natural numbers

1. $n^+ \neq \emptyset$ for all natural numbers n
2. Let n and m be natural numbers. If $n^+ = m^+$ then $n = m$

For 1. $\emptyset$ has no elements but n^+ always has at least one element

we'll show 2. on Monday once we have induction in this context.

2. An infinite set

So far (see homework 1) we don't have any infinite sets. In order to guarantee one we need to build it into our axioms

Axiom of infinity

as an element   as an element

There exists a set containing $\emptyset$ and containing the successor of each of its elements

That is there exists a set S with

- $0 \in S$
- for all $a \in S$, $\underset{a \cup \{a\}}{\overset{\shortparallel}{a^+}} \in S$

lower case omega

the smallest such set is called ω or the set of natural numbers.

But wait how do I know there is a smallest one? Have to show it

To answer these questions

Say a set A is a successor set if $0 \in A$ and $x^+ \in A$ wherever $x \in A$

The axiom gives us one successor set. Call it S

let $\mathcal{C} = \{ A \in \mathcal{P}(S) \mid A$ is a successor set $\}$

$S \in \mathcal{C}$ so $\mathcal{C}$ is nonempty

So I can take $\bigcap \mathcal{C}$

Claim $\bigcap \mathcal{C}$ is a successor set

- $0 \in \bigcap \mathcal{C}$ because 0 is an element of every successor set so 0 is an element of every element of $\mathcal{C}$

- take $x \in \bigcap \mathcal{C}$ want to show $\underbrace{x^+ \in \bigcap \mathcal{C}}$
  so $x \in A$ for every $A \in \mathcal{C}$
  But every $A \in \mathcal{C}$ is a successor set
  so $\underbrace{x^+ \in A \text{ for every } A \in \mathcal{C}}$
  so $x^+ \in \bigcap \mathcal{C}$

So $\bigcap \mathcal{C}$ is a successor set.

We want to show $\bigcap \mathcal{C}$ is the minimal successor set.
To do that choose any successor set B
(B does not have to relate to S in any way)

Note $S \cap B$ is a successor set by the same argument.

ie $0 \in S$ and $0 \in B$ so $0 \in S \cap B$

if $x \in S \cap B$ then $x \in S$ and $x \in B$

so $x^+ \in S$ and $x^+ \in B$

so $x^+ \in S \cap B$

and $S \cap B \subseteq S$ so $S \cap B \in \mathcal{C}$

so $\bigcap \mathcal{C} \subseteq S \cap B \subseteq B$ So $\bigcap \mathcal{C}$ is a successor set which is a subset of every successor set

so $\bigcap \mathcal{C} = \omega$

An example of a successor set is $\omega^+ = \omega \cup \{\omega\}$ and clearly $\omega \subseteq \omega^+$

does $\omega^+ = \omega$? This will happen if $\omega \in \omega$

No

Assume there is a successor set S with $\omega \not\subseteq S$

so there is some $n \in \omega$ $n \notin S$. Pick n smallest

$n \neq 0$ as by def $0 \in S$ so $(n-1) \in S$

but if n is a natural number (in our usual sense)

then $n = (n-1)^+$

but S is a successor set so $\underset{\overset{\shortparallel}{n}}{(n-1)^+} \in S$

contradiction.

For the break

Why is $\omega \subseteq S$ for every successor set S

Is $\omega = \{0, 1, 2, \ldots\}$?

Is $\omega = \{0, 1, 2, \ldots\}$

yes First check $\{0, 1, 2, \ldots\}$ is a successor set

yes $0 \in \{0, 1 \ldots\}$ and if $n \in \{0, \ldots\}$

then $n^+ = n+1 \in \{0, \ldots\}$

(that's what the ... means)

So $\omega \subseteq \{0, 1, 2, \ldots\}$

but if there was an $n \in \{0, 1, 2, \ldots\}$, but $n \notin \omega$. Take n minimal. Then $(n-1) \in \omega$ so $(n-1)^+ = n \in \omega$, contradiction

In fact we can take ω as the rigorous definition of $\{0, 1, 2, \ldots\}$

Historically people took a long time to accept infinite sets.

To see the problem, consider some process that gives an infinite sequence

$$0, 1, 2, \ldots$$

Thinking of this as a process, at any stage you only have finitely many things

$$\boxed{0} \quad \boxed{0,1} \quad \boxed{0,1,2} \quad \ldots$$

(cf. Aristotle's potential infinite)

Wheras in ω we have a whole infinity all at once

(cf Aristotle's actual infinite)

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ω is a place we can hide Russell's paradox

Another

let B be the set of interesting natural numbers.

let n be the smallest natural number which is not interesting. Well that's at least a mildly interesting property so $n \in B$, contradiction.

The joy of sets

2. Berry's paradox gets its name from G. G. Berry, an Oxford University librarian who communicated it to Russell. It concerns "the smallest integer that cannot be expressed in less than thirteen words." Since this expression has 12 words, to which set does the integer it describes belong: the set of integers that can be expressed in English with less than 13 words, or the set of integers that can be expressed only with 13 words or more? Either answer leads to a contradiction.

3. The philosopher Max Black expressed the Berry paradox in a fashion similar to the following version: Various integers are mentioned in this book. Fix your attention on the smallest integer that is not referred to in any way in the book. Is there such an integer?

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③ Next time

using ω – induction