Math 343 lecture 2

① Counting sequences and generating functions

Def Let $\mathcal{A}$ be a combinatorial class. Define $A(x)$, the (ordinary) generating function of $\mathcal{A}$ to be

what does this mean?

We say marks

How does this definition make sense?

compare to $\sum_{a \in \mathcal{A}} x$ ← does that make sense?

why?

In generating functions, the question does this make sense always means

For the definition of a generating function we're ok because last class we defined

Def A combinatorial class $\mathcal{A}$ is a countable set with a size function satisfying

(a) The size of any element of $\mathcal{A}$ is a non-negative integer

(b) The number of elements of any given size is finite

This is important so lets call it a proposition

Proposition let $\mathcal{A}$ be a combinatorial class. Then

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

the proof is simply the calculation above

Note

eg let $\mathcal{W}$ be the set of binary strings

let the size of a string be its length

so $\mathcal{W}_1 =$ $w_1 =$

$\mathcal{W}_2 =$ $w_2 =$

what is w_n?

How would I prove that?

Generally

Returning to generating functions then

$W(x) =$

empty string

eg let $\mathcal{A} = \{a\}$ and let $|a| = 1$

Is this a combinatorial class?

What is $A(x)$?

What if $|a| = 2$ then what is $A(x)$?

What if $|a| = 0$ then what is $A(x)$?

Give me a combinatorial class $\mathcal{A}$ (ok if its silly) with generating function $A(x) = 1 + 2x + x^2$

eg Last time we had the example of the class of binary rooted trees $\mathcal{T}$.

What is $T(x)$?

By definition it is

but

What about empty?

so $|T| =$

so $T(x) =$

So from the decomposition of T we obtained

Now what?

But there's only one $T(x)$ not two so which sign?

2 Formal power series

Def Let $a_0, a_1, \ldots$ be a sequence of real numbers then

formal power series

The formal means

Another way to say this is

Def Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be formal power series

Then $A(x) = B(x)$ if and only if

This is another manifestation of formal.

Further the reason we have to define equality is because as formal objects they only do what we tell them to do.

However,

This is a good definition in a human sense

Def Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be formal power series. Then

$$A(x) + B(x) =$$

$$A(x)B(x) =$$

There are many other things we can define so that they end up just as you'd expect from calculus power series including

(but

We will discuss some of these when they come up. If you are impatient see the notes.

③ Coefficient extraction

The basic tools to get coefficients out of formal power series are

Proposition Let $A(x)$ and $B(x)$ be formal power series

(a) $[x^n]A(px) =$

(b) $[x^n]\, x^m A(x) =$

(c) $[x^n](A(x)+B(x)) =$

(d) $[x^n](A(x)B(x)) =$

note any real number

(e) Extended Binomial Theorem

For n a positive integer and r a real number

$[x^n](1+x)^r =$

Note in (e) if r is also a positive integer then eg We had for binary rooted trees that

$$T(x) = \frac{1-2x - \sqrt{1-4x}}{2x}$$

So use the extended binomial theorem to expand $(1-4x)^{\frac{1}{2}}$

$[x^n](1-4x)^{\frac{1}{2}} =$

so $T(x) =$

So for every $n \geq 1$ there are

Let us collect two important things from this example

Proposition $\binom{\frac{1}{2}}{n} = \left(-\frac{1}{4}\right)^{n-1} \frac{1}{2n} \binom{2n-2}{n-1}$

The proof is the calculation above

Def The n^{th} Catalan number is $C_n = \frac{1}{n+1}\binom{2n}{n}$

These count many things (including binary rooted trees)

The Catalan sequence begins

Now lets get back to the coefficient extraction proposition since we havent talked about the proof yet

let $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $B(x) = \sum_{n=0}^{\infty} b_n x^n$

(a) $[x^n] A(px) = p^n [x^n] A(x)$

we didnt actually define composition but this is how you do it

proof of (a)

(b) $[x^n] \, x^m A(x) = [x^{n-m}] A(x)$

proof of (b)

(c) $[x^n](A(x)+B(x)) = [x^n]A(x) + [x^n]B(x)$

proof of (c)

(d) $[x^n](A(x)B(x)) = \sum_{k=0}^{n}([x^k]A(x))([x^{n-k}]B(x))$

proof of (d)

(e) $[x^n](1+x)^r = \binom{r}{n} = \frac{r(r-1)(r-2)\dots(r-k+1)}{k!}$

The proof of (e) is more subtle for two reasons

first

second

For now lets forget about the second difficulty and just prove the result for calculus power series

By Taylor series $[x^n] f(x) = \dfrac{f^{(n)}(0)}{n!}$

so

④ Next time

The partial fractions algorithm

Generally please read the notes before class (read the first two if you haven't yet)