MATH 817 ASSIGNMENT 3 SOLUTIONS

(1) Proof by induction. First note that $G = G^1$, $H = H^1$ and ϕ is surjective so $\phi(G^1) = H^1$. Suppose $n > 1$ and suppose inductively that $\phi(G^{n-1}) = H^{n-1}$. Then $\phi(G^n) = \phi([G^{n-1}, G]) = [\phi(G^{n-1}), \phi(G)] = [H^{n-1}, H] = H^n$.

(2) Since G is finite and nilpotent we have

$$1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = G \tag{1}$$

with $N_i \lhd G$ and $N_{i+1}/N_i \subseteq Z(G/N_i)$. Let N_{i+1}/N_i be a non-cyclic factor. Since N_{i+1}/N_i is finite abelian we can write $N_{i+1}/N_i = \prod_{j=1}^{m_i} C_{i,j}$ where $C_{i,j}$ is cyclic. Then selecting just one of the cyclic factors we have

$$N_{i+1}/N_i = C \times H$$

C and H both nontrivial, C cyclic. By the correspondence theorem define K by $H = K/N_i$. Then $N_i \subseteq K \subseteq N_{i+1}$. Consider the series

$$1 = N_0 \subseteq N_1 \subseteq \cdots N_i \subseteq K \subseteq N_{i+1} \subseteq \cdots \subseteq N_n = G \tag{2}$$

$K/N_i = H \subseteq Z(G/N_i)$ and $N_{i+1}/K \cong (N_{i+1}/N_i)/(K/N_i) \cong CH/H \cong C/(C \cap H) = C$. So all non-cyclic factors of (2) are central, and the sum of the sizes of all non-cyclic factors is smaller than in (1). Finally since $H = K/N_i \subseteq Z(G/N_i)$ we have for any $g \in G$ and $k \in K$ that $k^{-1}g^{-1}kg \in N_i$ so $g^{-1}kg \in N_iK = K$ so $K \lhd G$. Continuing inductively we get that G is supersolvable.

(3) Let G be supersolvable, so we have

$$1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = G$$

with the $N_i \lhd G$ and N_{i+1}/N_i cyclic. Proceed by induction on n.

Let M be a minimal normal subgroup of G. If $M \subseteq N_1 = N_1/N_0$ then M is cyclic.

Suppose $M \not\subseteq N_1$. Consider $H = M \cap N_1$. Take $g \in G$. Since $M \lhd G$, $H^g \subseteq M$, and since $N_1 \lhd G$, $H^g \subseteq N_1$. Thus $H \lhd G$. But M is minimal, so $H = 1$.

Let $\pi : G \to G/N_1$ be the canonical homomorphism. G/N_1 is supersolvable with series

$$1 = N_1/N_1 \subseteq N_2/N_1 \subseteq \cdots \subseteq N_n/N_1 = G/N_1$$

(since $(N_i/N_1)/(N_{i+1}/N_1) \cong N_i/N_{i+1}$) which has smaller length than the series of G.

$\pi(M)$ is a nontrivial normal subgroup of G/N_1. Suppose $K/N_1 \subsetneq \pi(M)$ were a nontrivial normal subgroup of G/N_1. Then by the correspondence theorem $M \lhd G$ and so $M \cap K \lhd G$. But $\pi(M \cap K) = K/N_1 \neq 1$ so $M \cap K \neq 1$ contradicting the minimality of M.

Thus $\pi(M)$ is a minimal nontrivial normal subgroup of G/N_1, and so by induction $\pi(M)$ is cyclic. But since $M \cap N_1 = 1$, π is an isomorphism on $M \to \pi(M)$ and so $\pi(M)$ is cyclic.

(4)

$[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x$
$= [x^{-1}yxy^{-1}, z]^y [y^{-1}zyz^{-1}, x]^z [z^{-1}xzx^{-1}, y]^x$
$= (yx^{-1}y^{-1}xz^1x^{-1}yxy^{-1}z)^y (zy^{-1}z^{-1}yx^{-1}y^{-1}zyz^{-1}x)^z (xz^{-1}x^{-1}zy^{-1}z^{-1}xzx^{-1}y)^x$
$= y^{-1}yx^{-1}y^{-1}xz^1x^{-1}yxy^{-1}zyz^{-1}zy^{-1}z^{-1}yx^{-1}y^{-1}zyz^{-1}xzx^{-1}xz^{-1}x^{-1}zy^{-1}z^{-1}xzx^{-1}yx$
$= 1$

(5) (a) Let $|G| = 4$. Either G is cyclic (hence solvable) or $x^2 = 1$ for all $x \in G$, so by a previous homework problem G is abelian, hence solvable.

(b) Let $|G| = pq$, $p > q$. Then we know G has a normal Sylow-p-subgroup, P. Then

$$1 \subseteq P \subseteq G$$

And $|P/1| = |P| = p$, so $P/1$ is cyclic since of prime order, and $|G/P| = q$ so G/P is cyclic since of prime order. Thus G is solvable.

(c) Let $|G| = 12 = 3 \cdot 2^2$. Then we know that G has a normal Sylow-3-subgroup, or a normal Sylow-2-subgroup. Call this group S. Then G/S has order 3 or 4 and S also has order 3 or 4. By the first part any group of order 4 is solvable and any group of order 3 is solvable since it is cyclic, hence abelian. Thus (Isaacs Corollary 8.4) G is solvable.

(d) Let $|G| = 36$. Let H be a Sylow-3-subgroup. Then $|G : H| = 4$. Consider the homomorphism $\phi : G \to S_4 \cong Sym(G/H)$ given by $g \mapsto (Hx \mapsto Hxg)$. Then $\ker(\phi) \subseteq H$ so $|\ker(\phi)| = 1$ or 3 or 9 and $|G|/|\ker(\phi)|$ divides $|S_4| = 24 = 3 \cdot 2^3$. So 3 divides $|\ker(\phi)|$. So $\ker(\phi)$ is nontrivial and has order 3 or 9.
This gives that $G/\ker(\phi)$ has order 4 or 12 and hence is solvable by previous parts. If $\ker(\phi)$ has order 3 it is abelian hence solvable. Finally consider $|\ker(\phi)| = 9$. Then $\ker(\phi)$ is a finite p-group and so has a nontrivial center. Thus either it equals its center and so is abelian hence solvable, or its center has order 3 giving a sequence which shows that $\ker(\phi)$ is solvable. In any case by Isaacs Corollary 8.4 G is solvable.