MATH 817 ASSIGNMENT 4

DUE NOVEMBER 5, 2009, IN CLASS

If your assignment must be late for any reason please notify me (by email, phone or in person) **before** the assignment is due. There will be no retroactive lates.

(1) (Rotman problem 7.29 iii, iv, v) Let A and B be (fixed) objects in a category $\mathcal{C}$. Define a new category $\mathcal{C}'$ whose objects are diagrams

$$A \xrightarrow{\alpha} C \xleftarrow{\beta} B$$

where C is an object in $\mathcal{C}$ and α and β are morphisms in $\mathcal{C}$. Define a morphism in $\mathcal{C}'$ to be a morphism θ in $\mathcal{C}$ that makes the following diagram commute

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & C & \xleftarrow{\beta} & B \\ \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{id}_B} \\ A & \xrightarrow{\alpha'} & C' & \xleftarrow{\beta'} & B \end{array}$$

There is an obvious candidate for composition and identity morphisms.

(a) Prove that $\mathcal{C}'$ is a category.

(b) Prove that an initial object in $\mathcal{C}'$ is a coproduct in $\mathcal{C}$.

(c) Give an analogous construction showing that a product is a terminal object in a suitable category.

(2) Show that pullbacks exist in the category of abelian groups. *Hint: Let A, B, C be abelian groups with homomorphisms $\phi : A \to C$, $\psi : B \to C$. Build the pullback as pairs (a, b) with $a \in A$, $b \in B$, and with certain restrictions on a and b.*

(3) (Rotman problem 7.38) Prove that there is a functor from the category of groups to the category of abelian groups taking G to G/G'.

(4) (Joy of Cats definition 6.9 and example 6.11.1b) A functor $F : \mathcal{A} \to Set$ is called *representable* (by an $\mathcal{A}$-object A) provided that F is naturally isomorphic to the hom-functor $Hom(A, -) : \mathcal{A} \to Set$, i.e. provided that there is a natural transformation $\tau : F \to Hom(A, -)$ where each τ_A is an isomorphism. Show that the forgetful functor from groups to sets is represented by $\mathbb{Z}$.

(5) (Isaacs problem 10.2) Let $G = D_8 \times Z_2$ where D_8 is the dihedral group of order 8 and Z_2 is cyclic of order 2. Let $X = \{x\}$ be a singleton set. Show how to make G into an X-group so that $Z(G)$ is not an X-subgroup.