MATH 817 ASSIGNMENT 6 SOLUTIONS

(1) Note first that eRe is a ring with multiplicative identity $e1e = e^2 = e$. Note also that for any $x \in eRe$, $exe = x$.

Take any $x \in eJ(R)e$. Then $x = ere$ where $r \in J(R)$. Let I be the ideal generated by x in eRe. Take any element $xs \in I$ where $s \in eRe$. $J(R)$ is an ideal so $xs = eres \in J(R)$, so xs is quasiregular in R. Let u be a unit of R such that $(1-xs)u = 1$. Then $(e-xs)(eue) = eue - xseue = eue - exseeue = eue - eexseue = eue - exsue = e(1-xs)ue = e1e = e$. So all elements of I are quasiregular in eRe, so $x \in I \subseteq J(eRe)$. Thus $eJ(R)e \subseteq J(eRe)$.

On the other hand let U be a simple right R-module. Then Ue is a right eRe-module. If $Ue \neq 0$ then take $ue \in Ue$, $u \in U$. Then $ueR = U$ by simplicity of U so $ueRe = Ue$. Thus Ue is generated by any of its nonzero elements and so it is also simple. Suppose $x \in J(eRe)$. Then $x = ere$ with $r \in R$. x annihilates every simple right eRe-module so $Uex = 0$. Thus $ex \in J(R)$ so $x = exe \in eJ(R)e$ giving $J(eRe) \subseteq eJ(R)e$.

(2) Let $I \subseteq J(R)$ be a right ideal. Let K be another right ideal such that $K + I = R$. Then we can write $1 = k + i$ with $k \in K$ and $i \in I$. But i is quasiregular so $k = 1 - i$ is a unit. Thus $K = R$. So I is small.

Let I be small. Take any $i \in I$. Suppose i is not quasiregular. Then $K = \langle 1 - i \rangle$ is a proper ideal of R but $K + I = R$ which is a contradiction. Thus all elements of I are quasiregular and so $I \subseteq J(R)$.

(3) (a) $\lambda - a = \lambda(1 - \lambda^{-1}a)$. $\lambda^{-1}a \in J(\mathbb{C}[G])$ hence is quasiregular. Thus $\lambda - a$ is a product of units and hence a unit itself.

(b) G is a basis for $\mathbb{C}[G]$ considered as a vector space over $\mathbb{C}$. The elements of $\mathcal{S}$ are distinct and so $\mathcal{S}$ is uncountable, but $\mathbb{C}[G]$ has a countable basis, thus $\mathcal{S}$ is linearly dependent.

(c) Take $a \in J(\mathbb{C}[G])$ and form $\mathcal{S}$ as in the previous part. $\mathcal{S}$ is linearly dependent so

$$\sum_{i=1}^{n} \frac{1}{\lambda_i - a} = 0$$

finding a common denominator we get a polynomial $P(a)$ of degree $n-1$ such that

$$\frac{P(a)}{\prod_{i=1}^{n} \lambda_i - a} = 0$$

Thus $P(a) = 0$ so a is algebraic over $\mathbb{C}$.

(d) Take $a \in J(\mathbb{C}[G])$. By the previous part a is algebraic over $\mathbb{C}$. So there is some polynomial P over $\mathbb{C}$, such that $P(a) = 0$. Note that the constant term of P must be in $J(\mathbb{C}[G])$ since $J(\mathbb{C}[G])$ is an ideal. Thus since $\mathbb{C}$ is a field and the $J(\mathbb{C}[G])$ is proper, P has zero constant term. So write

$$0 = P(a) = ca^k Q(a)$$

where Q is a polynomial with constant term 1 and $c \in \mathbb{C} \setminus \{0\}$. But then $Q(a) = (-x) + 1$ for some $x \in J(\mathbb{C}[G])$ so $Q(a)$ is a unit and so multiplying on the right by its inverse we get $0 = ca^k$, but c is also a unit, so $0 = a^k$. Thus $J(\mathbb{C}[G])$ is nil.

(e) Take $a \in J(\mathbb{C}[G])$. Then $a = \sum_{i=1}^{n} c_i g_i$ for some finite sum. Let $H = \langle g_1, \ldots, g_n \rangle$. The ideal A_G generated by a in $\mathbb{C}[G]$ has all elements quasiregular (since it is inside $J(\mathbb{C}[G])$). Thus every element in the the ideal A_H generated by a in $\mathbb{C}[H]$ is quasiregular in $\mathbb{C}[G]$).
Moreover the inverses can be chosen in $J(\mathbb{C}[H])$ because if $(1 - ar)u = 1$ with $r \in \mathbb{C}[H]$, then choose a set S of coset representatives of G/H with 1 representing H (transversal) and write $u = u_1 s_1 + \cdots + u_k s_k$ with $u_i \in \mathbb{C}[H]$ and $s_i \in S$. But then $\sum(1 - ar)u_i s_i = 1$ with $(1 - ar)u_i \in \mathbb{C}[H]$, so by the disjointness of cosets we see that exactly one s_i is nonzero and that one must be 1 representing H, giving $(1 - ar)s_i = 1$ for some i.
Thus every element in the the ideal A_H generated by a in $\mathbb{C}[H]$ is also quasiregular in $\mathbb{C}[H]$. Hence $a \in A_H \subseteq J(\mathbb{C}[H])$. But by the previous part $J(\mathbb{C}[H])$ is nil, so a is nilpotent, so $J(\mathbb{C}[G])$ is nil.

(f) Take $0 \neq \alpha \in J(\mathbb{C}[G])$. Write $\alpha = \sum c_g g$ with at least one $c_g \neq 0$. Then $\alpha\alpha^*$ is nonzero since the coefficient of 1 is $\sum c_g \bar{c}_g > 0$. But $(\alpha\alpha^*)^* = (\alpha^*)^*\alpha^* = \alpha\alpha^*$. Let $\beta = \alpha\alpha^*$, then $\beta^2 = \beta\beta^* \neq 0$ and $(\beta^2)^* = \beta^2$. Continuing likewise $\beta^4 \neq 0 \ldots \beta^{2^k} \neq 0$. So β is not nilpotent.

(g) Take $0 \neq \alpha \in J(\mathbb{C}[G])$. $\alpha\alpha^* \in J(\mathbb{C}[G])$ since $J(\mathbb{C}[G])$ is an ideal. By the previous part this element is not nilpotent, but by the part before $J(\mathbb{C}[G])$ is nil. Contradiction. Thus $J(\mathbb{C}[G]) = 0$.

(4) By the previous part we know that $\mathbb{C}[S_3]$ is a quasiregular ring, and since it is a finite dimensional algebra it is right artinian. Thus $\mathbb{C}[S_3]$ is wedderburn, and so it must be a sum of full matrix rings over $\mathbb{C}$. $|S_3| = 6$, and 6 can be written as a sum of squares in the following ways: $6 = 1 + 1 + 4, 6 = 1 + 1 + 1 + 1 + 1 + 1$ but S_3 is not abelian, so $\mathbb{C}[S_3] \not\cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Thus $\mathbb{C}[S_3] \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$.

(5) $D_{12} = \langle a, b | a^6 = b^2 = 1, bab = a^5 \rangle$. Then $|D_1 2| = 12$ and, by calculation, $D'_{12} = \langle a^2 | a^6 = 1 \rangle$. Thus there are $12/3 = 4$ linear characters and so, to make 12 as the sum of the squares of the orders, the orders of the characters must be $1, 1, 1, 1, 2, 2$.

Take a^ℓ. The conjugates of a^ℓ are $b^\epsilon a^{-k} a^\ell b^\epsilon a^k$ which is a^ℓ if $\epsilon = 0$ and $a^{-\ell}$ if $\epsilon = 1$. Likewise compute that the conjugates of $a^\ell b$ are $a^{\ell - 2k} b$ and $a^{2k - \ell} b$ for any integer k. Thus representatives of the conjugacy classes of D_{12} are $1, a, a^2, a^3, b, ab$. So far we know

	1	a	a^2	a^3	b	ab
χ_1	1	1	1	1	1	1
χ_2	1		1			
χ_3	1		1			
χ_4	1		1			
χ_5	2					
χ_6	2					

Consider D_{12}/D'_{12} in more detail. $D_{12}/D'_{12} = \langle a, b | a^2 = b^2 = 1, ab = ba \rangle$. So it is isomorphic to the direct product of two cyclic groups of order 2 which has characters the four different choices of ± 1 on each factor. Thus

	1	a	a^2	a^3	b	ab
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	-1
χ_3	1	1	1	1	-1	-1
χ_4	1	-1	1	-1	-1	1
χ_5	2					
χ_6	2					

Now use orthogonality. Fill in the table

	1	a	a^2	a^3	b	ab
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	-1
χ_3	1	1	1	1	-1	-1
χ_4	1	-1	1	-1	-1	1
χ_5	2	q	r	s	t	u
χ_6	2	v	w	x	y	z

Get the system of linear equations

$$\begin{aligned}
2+2q+2r+s+3t+3u&=0\\
2-2q+2r-s+3t-3u&=0\\
2+2q+2r+s-3t-3u&=0\\
2-2q+2r-s-3t+3u&=0\\
2+2v+2w+x+3y+3z&=0\\
2-2v+2w-x+3y-3z&=0\\
2+2v+2w+x-3y-3z&=0\\
2-2v+2w-x-3y+3z&=0
\end{aligned}$$

and the nonlinear equations

$$\begin{aligned}
4+2q^2+2r^2+s^2+3t^2+3u^2&=12\\
4+2v^2+2w^2+x^2+3y^2+3z^2&=12\\
4+2qv+2rw+sx+3ty+3uz&=0
\end{aligned}$$

Solving just the linear part gives

$$w=r=-1, t=u=y=z=0, s=-2q, x=-2v$$

so the nonlinear part becomes

$$\begin{aligned}
6+6q^2&=12\\
6+6v^2&=12\\
6+6qv&=0
\end{aligned}$$

so $qv = -1$, $q^2 = 1$, $v^2 = 1$, giving the final table

	1	a	a^2	a^3	b	ab
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	-1
χ_3	1	1	1	1	-1	-1
χ_4	1	-1	1	-1	-1	1
χ_5	2	1	-1	-2	0	0
χ_6	2	-1	-1	2	0	0