LANGUAGE OF SCHEMES

KAREN YEATS

Unless otherwise stated, all rings considered here are commutative with multiplicative unit

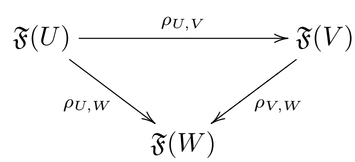
1. Sheaves

Definition 1. [Sheaves] Let X be a topological space. We say $\mathfrak{F}$ is a *sheaf of rings* over X, if for every open set U of X, we are given a ring $\mathfrak{F}(U)$ (called ring of sections), and for every inclusion of open sets $V \subset U$, we are given ring homomorphism (called restriction)

$$\rho_{U,V} : \mathfrak{F}(U) \longrightarrow \mathfrak{F}(V)$$

such that the following conditions are satisfied:

(1) For $W \subset V \subset U$, open sets in X, the following diagram commutes:



(2) Let U be an open set in X and $(U_i)_{i\in I}$ be an open covering of U. Assume that we are given a family of elements $(s_i \in \mathfrak{F}(U_i))_{i\in I}$ such that, for every pair of indices i, j, we have

$$\rho_{U_i,U_i\cap U_j}(s_i) = \rho_{U_j,U_i,U_j}(s_j)$$

then there is a unique section $s \in \mathfrak{F}(U)$, such that $\rho_{U,U_i}(s) = s_i$.

Remark 1.
In the definition of a sheaf, if we drop last condition, we get a presheaf. A presheaf can also be defined as follows: Let $\mathcal{C}$ be a category, whose objects are open sets in X and for any pair of open sets U, V in X, $Hom_{\mathcal{C}}(U, V)$ is a singleton, if $U \subset V$ and empty set otherwise. A presheaf $\mathfrak{F}$ on X with values in category $\mathcal{D}$ (category of commutative rings in Definition **??**) is a contravariant functor $\mathcal{C} \longrightarrow \mathcal{D}$

Remark 2. For the sake of convenience, we use the following notations: if $V \subset U$ and $s \in \mathfrak{F}(U)$, we write $s|_V$ for $\rho_{U,V}(s)$ (section s restricted to V). Also, it is sometimes convenient to write $\Gamma(U, \mathfrak{F})$ instead of $\mathfrak{F}(U)$ (ring of sections over open set U).

Definition 2. Let X be a topological space and $\mathfrak{F}$, $\mathfrak{G}$ be two sheaves of rings on X. A *morphism* $\varphi : \mathfrak{F} \to \mathfrak{G}$ between two sheaves, is a collection of ring homomorphisms $\varphi_U : \mathfrak{F}(U) \to \mathfrak{G}(U)$, for every open set U in X, such that for every pair U, V of open sets, with $V \subset U$, we have following commutative diagram:

$$\begin{array}{ccc} \mathfrak{F}(U) & \xrightarrow{\varphi_U} & \mathfrak{G}(U) \\ \downarrow \rho^{\mathfrak{F}U,V} & & \downarrow \rho^{\mathfrak{G}}_{U,V} \\ \mathfrak{F}(V) & \xrightarrow{\varphi_V} & \mathfrak{G}(V) \end{array}$$

Now let X be a topological space and $\mathfrak{F}$ be sheaf of rings over X. Let $x \in X$ be any point. The family of open sets in X containing x form a directed system, with respect to inclusions and thus family $\{\mathfrak{F}(U)\}_{x \in U}$, where U are open in X, is a directed family. We define $\mathfrak{F}_x := \varinjlim \mathfrak{F}(U)$, called *stalks of* $\mathfrak{F}$ *over* x.

Remark 3. The property of exactness in category of sheaves is a local property. In other words, if X is a topological space and $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ are three sheaves over X, then the sequence of morphisms

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow 0$$

is exact if and only if for every $x \in X$, the following sequence, induced by these morphisms, is exact:

$$0 \longrightarrow \mathfrak{F}_x \longrightarrow \mathfrak{G}_x \longrightarrow \mathfrak{H}_x \longrightarrow 0$$

Example 1. Let X, Y be two topological spaces. For any open set $U \subseteq X$, we associate $\mathfrak{F}(U)$, being set of continuous maps $U \to Y$. This is automatically a sheaf, since being continuous is *local property.*

Caution 1. One needs to be careful about defining *image* and cokernel of morphism of sheaves. Reason being *surjectivity* of $\varphi : \mathfrak{F} \to \mathfrak{G}$ does not mean that for every open set U of X, $\varphi_U : \mathfrak{F}(U) \to \mathfrak{G}(U)$ is surjective. It only means that given any element $s \in \mathfrak{G}(U)$, we can find a covering $U = \cup_i U_i$ such that each of $s|_{U_i}$ has a preimage in $\mathfrak{F}(U_i)$. We give important things to keep in mind:

(1) $Ker(\varphi)$ is the sheaf given by $U \longmapsto Ker(\varphi_U)$.
(2) $Im(\varphi)$ defined as $U \longmapsto Im(\varphi_U)$ is only a presheaf.
(3) $Coker(\varphi)$ defined as $U \longmapsto \mathfrak{G}(U)/Im(\varphi_U)$ is only a presheaf.

Thus, image and cokernel of morphism between sheaves is obtained by *sheafification* of presheaves defined in last two points above.

2. Affine Schemes

Let R be a commutative ring with identity.

Definition 3. *Spec*(R) is defined to be (as a set) collection of prime ideals in R, together with *Zariski topology*, where closed sets are of the form:

$$V(S) := \{\mathfrak{p} : \mathfrak{p} \text{is prime ideal in } R \text{ and } S \subset \mathfrak{p}\}$$

where S is any subset of R. Note that *distinguished open sets* in this topological space are of the form $Spec(R) \setminus V(f)$, which is same as $Spec(R_f)$, for $f \in R$. We denote such set by $Spec(R)_f$. Moreover, there is a sheaf of rings on $Spec(R)$, called *structure sheaf*, denoted by $\mathcal{O}_{Spec(R)}$, defined as follows:

$$\mathcal{O}_{Spec(R)}(Spec(R)_f) = R_f$$

$$\mathcal{O}_{Spec(R)}(U) = \varprojlim_{\substack{V \subset U \\ V,\ \text{distinguished open}}} \mathcal{O}_{Spec(R)}(V)$$

Remark 4. Note that $\mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) = R$ and $\mathcal{O}_{\mathrm{Spec}(R)}(\phi) = \{0\}$.

3. Schemes

Definition 4. A *scheme* is a topological space X together with a sheaf of rings (called *structure sheaf*) $\mathcal{O}_X$, such that we can find an open cover $X = \cup_i U_i$, where each U_i is affine (i.e, there exists ring R_i such that $(U_i, \mathcal{O}_X|_{U_i}) \cong (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$.

We can give definitions about schemes, which depend on the properties of rings involved.

Definition 5. A scheme X is called *locally noetherian* if we can find an open covering of X, where each open set is spectrum of a Noetherian ring. A scheme X is called reduced if for every open set U of X, the ring $\mathcal{O}_X(U)$ is reduced ring. (i.e, doesn't have any non-zero zero divisors). A scheme X is called integral if for every open set U of X, the ring $\mathcal{O}_X(U)$ is an integral domain.

Mostly, we will be interested in schemes over a field K. By a K-scheme, we mean a scheme X, such that the corresponding structure sheaf $\mathcal{O}_X$ is a sheaf of K-algebras (i.e, each $\mathcal{O}_X(U)$ is K-algebra and restriction maps are K-algebra homomorphisms.

Remark 5. The above definition can be equivalently rephrased by saying that *Spec*(K) *is terminal object in the category of K-schemes*, or that there is a distinguished morphism, called *structure morphism* $X \longrightarrow \mathrm{Spec}(K)$.

A scheme over K is said to be of finite type, if it can be covered by open sets U_i, where each $U_i = \mathrm{Spec}(A_i)$ and each A_i is finitely generated K-algebra. A schme X over K is said to be *finite* if it is affine ($X = \mathrm{Spec}(A)$), where A is finite dimensional as K vector space. A scheme X over K (of finite type) is called *étale* if $X = \cup_\alpha \mathrm{K}_\alpha$,where each K_α is étale over K. [1].

[1]that is, the module of differentials $\Omega_K(K_\alpha) = 0$