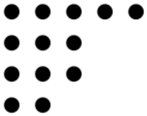
Plane partitions and tilings

Integrable Models

Sophie Burrill

February 23, 2011

Partition	Plane partition
13	28
	4 3 3 2 3 3 2 1 2 1 1 1 1 1

Introduction

4	3	3	2
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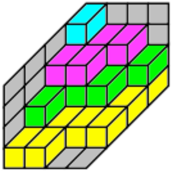
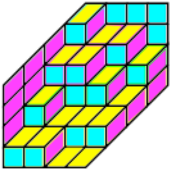


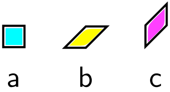
Introduction

- Plane partitions are another integrable model.
- Can be identified with (a special case of) the 6 vertex model.
- **Plane partitions=rhombus tilings of a hexagon.**

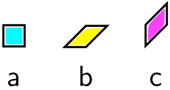
Plane partitions $\longrightarrow$ 6 vertex model?



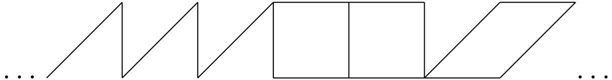
There are three types of blocks/tiles:



Plane partitions ⟶ 6 vertex model?



Any row in a plane partition is of the form:



which is . . . c c c a a b . . .

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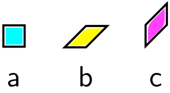

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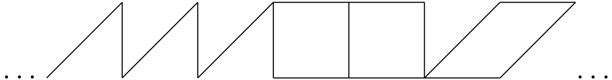


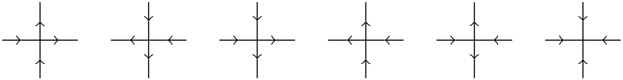
which is ... c c c a a b ...

A plane partition configuration is entirely determined by the presence of horizontal lines.

Plane partitions $\longrightarrow$ 6 vertex model?

a b c

Recall the 6 vertex model:



a_1 a_2 b_1 b_2 c_1 c_2

Plane partitions $\longrightarrow$ 6 vertex model?

a b c

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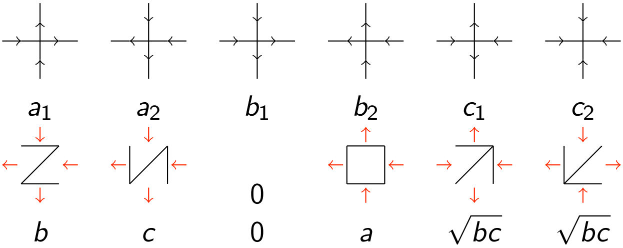

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- We see that this is actually a **five** vertex model.
- We **cannot** go from here to Alternating Sign Matrices, as there are different numbers of tiles in different rows.
- However, there are *subclasses* of plane partitions, one of which is conjectured to be in bijection with ASMs.

Plane partitions $\longrightarrow$ tilings

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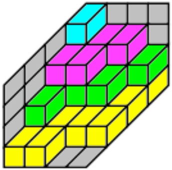

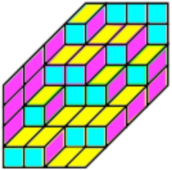

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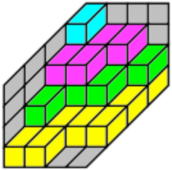

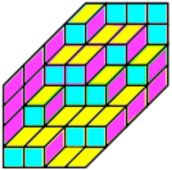

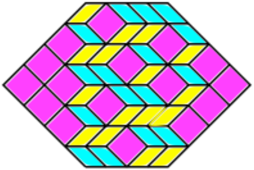

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    - We could not make use of the connection to the 6 vertex model, what other strategies will this new interpretation give?
2. Can we enumerate (and define!) ‘symmetric’ hexagons?

Answer 1:

Theorem

(MacMahon) The number of rhombus tilings of a hexagon with sides a, b, c, a, b, c *is*

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

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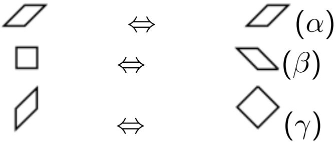
- 9 cases have symmetries;
- 8 of these have been enumerated;
- 1 case has an 'almost proof';

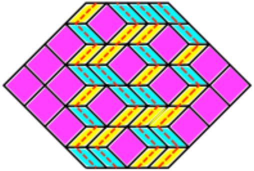
Preliminaries

First, formalize the 'straightening' that occurred between the plane partition and hexagon.



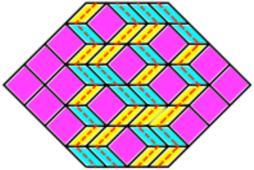
Non intersecting lattice paths

Consider the natural mapping between rhombus tilings of hexagons and non intersecting lattice paths



There are 4 paths from the bottom to the top of this hexagon through tiles of shape α and β.

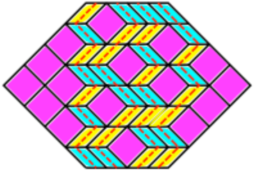
Non intersecting lattice paths

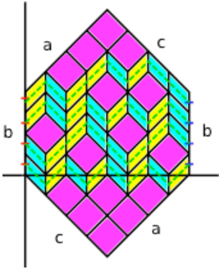


These non intersecting lattice paths **completely determine** the tiling of the hexagon of shape $a \times b \times c$!

Goal: Count the number of non intersecting paths on a hexagon of shape $a \times b \times c$.

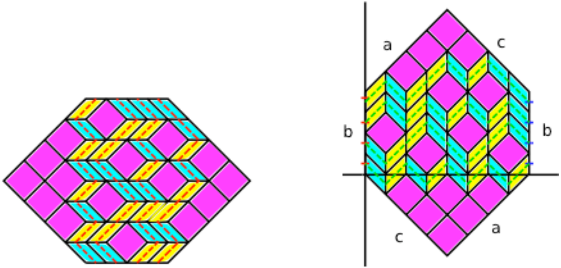
Non intersecting lattice paths





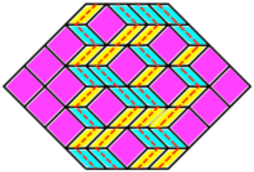
Steps $(1,1)$ and $(1,-1)$.

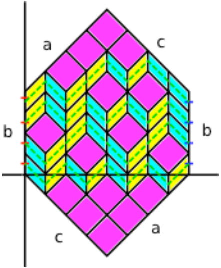
Non intersecting lattice paths



Here: how many ways to draw 4 non intersecting paths from $(0,1),(0,2),(0,3),(0,4)$ to $(8,1),(8,2),(8,3),(8,4)$?

Non intersecting lattice paths





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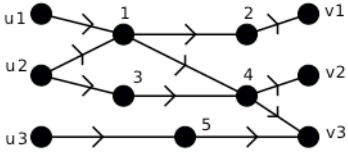
General: How many ways of drawing b paths from $(0,1),\ldots,(0,b)$ to $(a+c,1),\ldots,(a+c,b)$?

- Use the Lindstrom, Gessel-Viennot theorem that gives a method for finding non intersecting paths between two sets of vertices in a digraph through a determinant of all paths between two sets of vertices.

- D acyclic digraph



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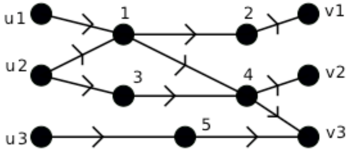
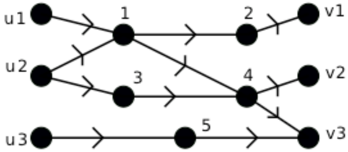
- k-vertex is k tuple of vertices;



- D acyclic digraph
- k-vertex is k tuple of vertices;
- $\mathbf{u}=(u_1, \ldots, u_k)$, $\mathbf{v}=(v_1, \ldots, v_k)$ k-vertices



- k-path $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ (where A_i is a path from u_i to v_i)

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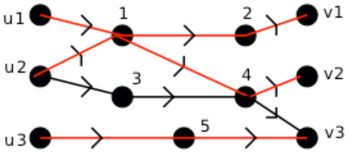

$$\mathbf{A}^* := (\{u1, 1, 2, v1\}, \{u2, 1, 4, v2\}, \{u3, 5, v3\})$$

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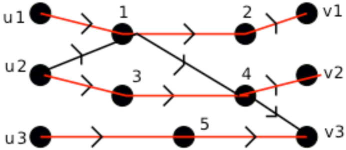

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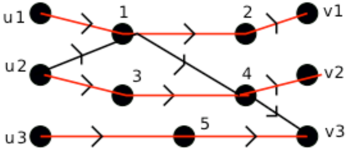

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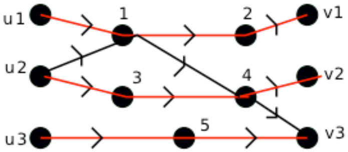
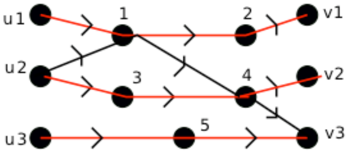
$\mathbf{A}^{**}$ is *disjoint* (non intersecting).

- Give weight to every edge;



For simplicity, in this example each edge gets weight 1.

- Give weight to every edge;
- Path weight:=product of edge weights;



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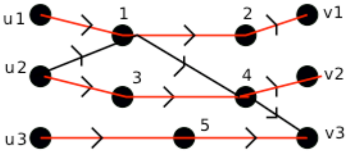
- k-path weight:=product of path weights



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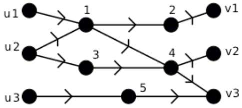
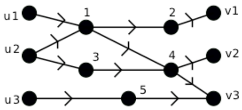


$P(u_i, v_j)$:=the set of paths from u_i to v_j
$P_w(u_i, v_j)$:= sum of their weights.



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1,1	$\{u1, 1, 2, v1\}$	1	2,3	$\{u2, 1, 4, v3\}$, $\{u2, 3, 4, v3\}$	2
1,2	$\{u1, 1, 4, v2\}$	1	3,1	$\emptyset$	0
1,3	$\{u1, 1, 4, v3\}$	1	3,2	$\emptyset$	0
2,1	$\{u2, 1, 2, v1\}$	1	3,3	$\{u3, 5, v3\}$	1
2,2	$\{u2, 1, 4, v2\}$, $\{u2, 3, 4, v2\}$	2			

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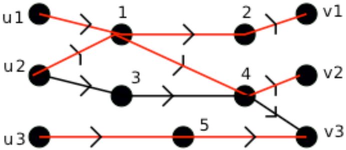

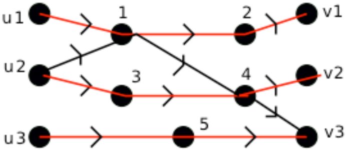

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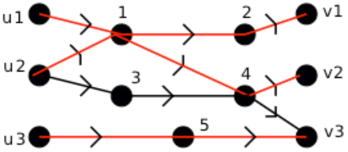

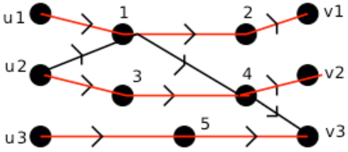

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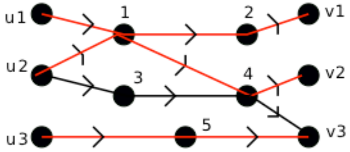

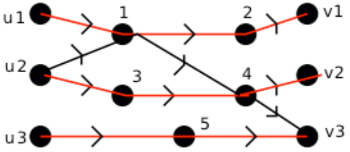

Example: $P(\mathbf{u}, \mathbf{v}) = \{\mathbf{A}^*, \mathbf{A}^{**}\}$, $P_w(\mathbf{u}, \mathbf{v}) = 2$.

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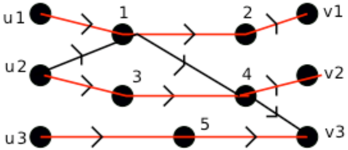

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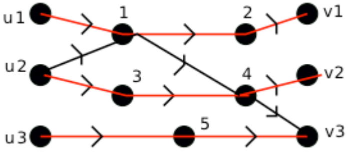

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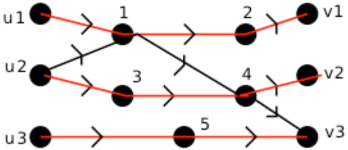

Example: $N(\mathbf{u},\mathbf{v}) = \{\mathbf{A}^{**}\}$, $N_w(\mathbf{u},\mathbf{v}) = 1$.

Theorem

(Lindstrom)

$$\sum_{\pi \in S_k} (sgn(\pi)) N(\mathbf{u}, \pi(\mathbf{v})) = \det_{1 \le i,j \le k} P(u_i, v_j)$$

($\pi(\mathbf{v})$ is the k-vertex $(v_{\pi(1)} \ldots, v_{\pi(k)})$)

$$\sum_{\pi \in S_k} (sgn(\pi)) N(\mathbf{u}, \pi(\mathbf{v})) = \det_{1 \leq i,j \leq k} P(u_i, v_j)$$

Example

$N(\mathbf{u}, \pi(\mathbf{v}))$=1 when $\pi = (123) \Rightarrow$ LHS=1.

$$RHS = \begin{vmatrix} P(u_1, v_1) & P(u_1, v_2) & P(u_1, v_3) \\ P(u_2, v_1) & P(u_2, v_2) & P(u_2, v_3) \\ P(u_3, v_1) & P(u_3, v_2) & P(u_3, v_3) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ \emptyset & \emptyset & 1 \end{vmatrix} = 1$$

Proof (sketch)

Key: nondisjoint k-paths will be 'cancelled out' through $sgn(\pi)$.

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Consider a nondisjoint k-path:
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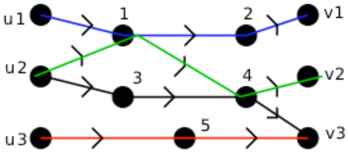
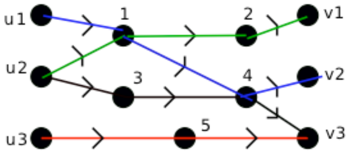


Proof continued

Create new paths at first point of intersection:
B^* :



$A^* \in P(\mathbf{u}, (123)\mathbf{v})$, $sgn(123) = 1$;
$B^* \in P(\mathbf{u}, (213)\mathbf{v})$, $sgn(213) = -1$.

This canceling reduces to give:

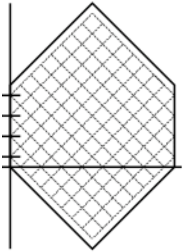
Assertion:

$$\sum_{\pi \in S_k} (sgn(\pi)) N(\mathbf{u}, \pi(\mathbf{v})) = \sum_{\pi \in S_k} (sgn(\pi)) P(\mathbf{u}, \pi(\mathbf{v})) \tag{1}$$

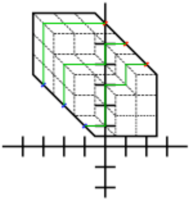
And RHS of (1) reduces to give original determinant.

Applicability?

Can be used for non intersecting lattice paths on rhombus tilings of hexagons: all steps are $(1,1)$ and $(1,-1)$ with edges having left to right orientation:

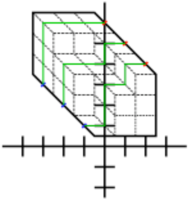


- If $a = c$: this is the number of such free Dyck paths between $(0,0)$ and $(0,2a)$, $\binom{2m}{m}$.
- Else, rotate again:

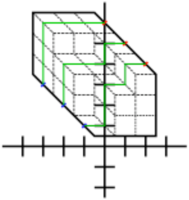


Starting vertices: $\mathbf{u} = (-1,1), (-2,2), \ldots, (-b,b)$
Ending vertices: $\mathbf{v} = (-1+a, 1+c), \ldots, (-b+a, b+c)$.

In general:



we are considering paths from $(-i, i)$ to $(-i + a, -i + c)$.
When $i = 0$, the number of such paths from $(0, 0)$ to (a, c) is $\binom{a+c}{c}$



Number non intersecting paths from side b to side b:

$$\det_{1\leq i,j\leq b}\left(\binom{a+c}{a-i+j}\right).$$

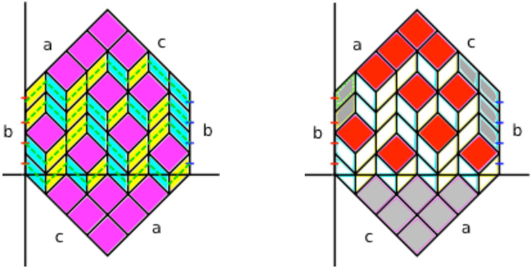
Where are we?

- This completes our goal of counting the number of non intersecting paths in a rhombus tiling of a hexagon of size $a \times b \times c$.

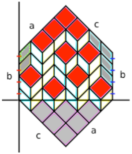
Where are we?

- This completes our goal of counting the number of non intersecting paths in a rhombus tiling of a hexagon of size $a \times b \times c$.
- If does **not** count the number of PPs of size n inside a box with sides $a \times b \times c$.

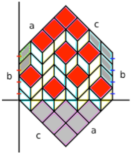
Count number PPs in hexagon according to size n of PP?

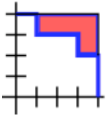


Map: $(1,1) \to (1,0)$; $(1,-1) \to (0,1)$.

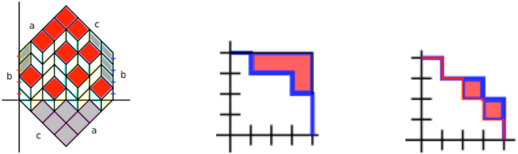


Map: $(1,1) \to (1,0)$; $(1,-1) \to (0,1)$.





Map: $(1,1) \rightarrow (1,0)$; $(1,-1) \rightarrow (0,1)$.



These are the first two paths in the example above.
We wish to count the are highlighted in pink.

Goal: Count the number of b non intersecting paths from $(0, b)$ to $(a, b-c)$ according to the area between the paths.

$$GF(paths(0,m) \to (n,0); q^{area}) = \left[\begin{array}{c} m+n \\ n \end{array} \right]_q$$

$$= \frac{1-q)(1-q^2)\dots(1-q^{m+n})}{(1-q)\dots(1-q^n)(1-q)\dots(1-q^m)}.$$

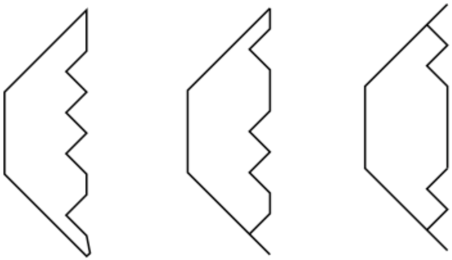
$[q^n]F(q)$:= no. plane partitions of size n in a hexagon of size $a \times b \times c$.

$$F(q) = \det_{1 \le i,j \le b} \left(q^{j(j-1)} \begin{bmatrix} a+c \\ a-i+j \end{bmatrix}_q \right)$$

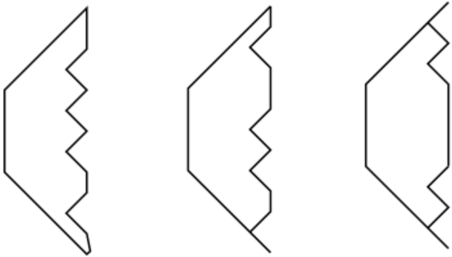
(Case 1: *unrestricted*)

For small n this is can be manageable, but extra determinant evaluation techniques such as *condensation* or *LU factorization* should be employed.

Symmetric PPs: invariant under reflection in vertical axis



Symmetric PPs: invariant under reflection in vertical axis



Counted by:

$$\det_{1\le i,j\le n}\left(\binom{2m+1}{m-i+j}+\binom{2m+1}{m-i-j+1}\right).$$

(Case 2)

Cyclic symmetric PPs: invariant under rotation of 120 degrees

Cyclic symmetric PPs: invariant under rotation of 120 degrees

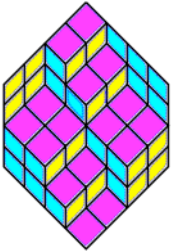
Counted by:

$$\det_{0\leq i,j,\leq n-1}\left(\delta_{i,j}+\binom{i+j}{i}\right)$$

$\delta_{i,j}$: sum of the principle minors. (Case 3)

Self complementary PPs: invariant under rotation by 180 degrees

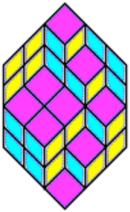
Rotate by 180 degrees:



Symmetric functions are used in enumeration.
(Case 5)

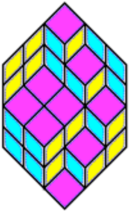
Transpose complementary PPs: the complement is equal to the mirror image

Reflect in horizontal axis:



Transpose complementary PPs: the complement is equal to the mirror image

Reflect in horizontal axis:



Counted by:

$$\det_{0 \le i,j \le n-1} \left(C_{i+j+a} \right)$$

Where C_i is the i^{th} Catalan number.
(Case 6)

Case	S.	CS	SC.	TC	Name
1					no restriction
2	x				SPP
3		x			CSPP
4	x	x			TSPP(*)
5			x		SCPP
6				x	TCPP
7	x		x		SSCPP
8		x		x	CSTCPP
9		x	x		CSSCPP
10	x	x	x		TSSCPP

(*)-‘almost proof’

A Theorem

Theorem: The number of TSSCPP os size $2n \times 2n \times 2n =$ the number of ASMs of size $n \times n$ (Zeilberger, then Kuperberg)

A Conjecture

Conjecture: There exists a simple, natural bijection between ASMs and TSSCPPs.

Thank you!

$$M_b^a = \left(\begin{array}{cccccc} & & & a & & \\ & * & * & | & * & * \\ & * & * & | & * & * \\ & * & * & | & * & * \\ b & -- & -- & | & -- & -- \\ & * & * & | & * & * \\ & * & * & | & * & * \end{array}\right)$$

$$M_b^a = \begin{pmatrix} & & & a & & \\ & * & * & | & * & * \\ & * & * & | & * & * \\ & * & * & | & * & * \\ b & -- & -- & | & -- & -- \\ & * & * & | & * & * \\ & * & * & | & * & * \end{pmatrix}$$

$$\det M = \frac{\det M_1^1 \det M_n^n - \det M_n^1 \det M_1^n}{\det M_{1,n}^{1,n}}$$

- $A = \left(\binom{a+c}{a-i+j}\right)_{1\leq i,j\leq b}$

- $A = \left(\binom{a+c}{a-i+j}\right)_{1\leq i,j\leq b}$
- $\det A_1^1 = \det_{2\leq i,j\leq n}\left(\binom{a+c}{a-i+j}\right) = \det_{1\leq i,j\leq n-1}\left(\binom{a+c}{a-i+j}\right) = A_n^n$

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- $\det A_1^n = A_n^1 = \det_{2\leq i,\leq n,1\leq j\leq n}\left(\binom{a+c}{a-i+j}\right) =$ $\det_{1\leq i,\leq n-1}\left(\binom{a+c}{a-1-i+j}\right)$
- $\det A = \det_{1\leq i,j\leq b}\left(\binom{a+c}{a-i+j}\right) = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$

(MacMahon's theorem)

LU factorization

- $A = \left(\binom{a+c}{a-i+j}\right)_{1\leq i,j\leq b}.$

LU factorization

- $A = \left(\binom{a+c}{a-i+j}\right)_{1\leq i,j\leq b}$. Solve through $L.U$ factorization?

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LU factorization

- $A = \left(\binom{a+c}{a-i+j}\right)_{1\leq i,j\leq b}$. Solve through $L.U$ factorization?
- Try for small $n = \{1, 2, 3, 4, \ldots\}$ to solve $M(n).U(n) = L(n)$
- Guess! Easy?

- Identification of factors

- Identification of factors
- Guessing (computer)

Next step?

Employ symmetric functions!