SPIN CHAINS IN COMBINATORICS

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ABSTRACT. We give a review of the elementary properties of spin chains together with some applications to the enumeration of alternating sign matrices. The main references concerning spin chains are [1, 2].

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1. Introduction

Spin chains are the simplest examples of the so called quantum integrable systems. The idea is to translate Liouville's definition of classical integrable systems in the formalism of quantum mechanics.

Definition 1.1. *A system is said to be **Liouville–integrable** if it is an Hamiltonian system such that the number of independent constants of the motion in involution is equal to the number of degrees of freedom.*

Hence a classical system formulated in terms of geometric objects is integrable if it has enough symmetries, their consequences being the existence of constants of the motion. When these conditions are satisfied the presence of these extra structures often gives the possibility to investigate the solutions in detail.

In quantum statistical mechanics a system is given by a Hilbert space called the **space of states** of the system and a selfadjoint operator acting on this space called the **Hamiltonian** of the system. The spectrum of this Hamiltonian corresponds to the set of values the energy of the system can take. Our problem is to compute this spectrum. For certain spin chains this is a reasonable goal if we extend the methods of classical integrability to an operatorial setting.

2. Necessary Conditions for the Integrability of Spin Chains

Let $N \in \mathbb{N}^*$, V and A be two complex vector spaces. The integer N fixes the size of the system, V is the local space of states and A is the auxiliary space.

The **space of states** of the system is the tensor product of N copies of the local space of states:

$$St \stackrel{\text{def}}{=} V^{\otimes N} = V \otimes \, ... \otimes V \otimes \, ... \otimes V \, ,$$

the n^{th} factor in the tensor product is called the n^{th} local space of states.

We introduce a series of operators which are caracterised as solutions of the necessary conditions for integrability:

- the **L-operators** $L_{a,n}(\lambda) \in \mathbf{E}nd(A \otimes St)$ for $n \in [\![1, N]\!]$ and $\lambda \in \mathbb{C}$. The index (a, n) indicates that the operator acts trivially everywhere except on the auxiliary space A and the n^{th} local space of states;

- the **R-operator** $R_{a_1,a_2}(\lambda_1, \lambda_2) \in \mathbf{E}nd(A \otimes A)$ with $\lambda_1, \lambda_2 \in \mathbb{C}$. The index (a_1, a_2) indicates that it acts nontrivially on the two copies of the auxiliary space. We assume that the R-operator is invertible;

Definition 2.1. *The **monodromy** of the system is the operator $T_a(\lambda) \in \mathbf{E}nd(A \otimes St)$ defined as the product of the L-operators on each site:*

$$T_a(\lambda) \stackrel{def}{=} L_{a,N}(\lambda) L_{a,N-1}(\lambda) \ldots L_{a,1}(\lambda) \, .$$

The R and L-operators are solution of the following necessary conditions for integrability:

- the **Yang–Baxter equation** in $\mathbf{E}nd(A^{\otimes 3})$ for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$

$$R_{a_1,a_2}(\lambda_1,\lambda_2)\, R_{a_1,a_3}(\lambda_1,\lambda_3)\, R_{a_2,a_3}(\lambda_1,\lambda_2) \;=\; R_{a_2,a_3}(\lambda_1,\lambda_2)\, R_{a_1,a_3}(\lambda_1,\lambda_3)\, R_{a_1,a_2}(\lambda_1,\lambda_2)\,;$$

- the **intertwining equation** in $\mathbf{E}nd(A^{\otimes 2}\otimes V)$ for $n \in [\![1,N]\!]$ and $\lambda_1, \lambda_2 \in \mathbb{C}$

$$R_{a_1,a_2}(\lambda_1,\lambda_2)\, L_{a_1,n}(\lambda_1)\, L_{a_2,n}(\lambda_2) \;=\; L_{a_2,n}(\lambda_2)\, L_{a_1,n}(\lambda_1)\, R_{a_1,a_2}(\lambda_1,\lambda_2)\,;$$

- the **ultralocality equation** in $\mathbf{E}nd(A^{\otimes 2}\otimes St)$ for $1 \leq n \neq m \leq N$ and $\lambda_1, \lambda_2 \in \mathbb{C}$

$$L_{a_1,n}(\lambda_1)\, L_{a_2,m}(\lambda_1) \;=\; L_{a_2,m}(\lambda_1)\, L_{a_1,n}(\lambda_1)\,.$$

From these relations we can deduce a few elementary properties indicating the possibility of integrability.

Theorem 2.2. *The monodromy of the system satisfies an intertwining equation in* $\mathbf{E}nd(A^{\otimes 2}\otimes St)$*:*

$$R_{a_1,a_2}(\lambda_1,\lambda_2)\, T_{a_1}(\lambda_1)\, T_{a_2}(\lambda_2) \;=\; T_{a_2}(\lambda_2)\, T_{a_1}(\lambda_1)\, R_{a_1,a_2}(\lambda_1,\lambda_2)\,.$$

Proof. It is a computation. We rewrite the left hand side of the equation in terms of the L-operators:

$$\begin{aligned} &R_{a_1,a_2}(\lambda_1,\lambda_2)\, L_{a_1,N}(\lambda_1) \dots L_{a_1,1}(\lambda_1)\, L_{a_2,N}(\lambda_2) \dots L_{a_2,1}(\lambda_2) \\ &\qquad = R_{a_1,a_2}(\lambda_1,\lambda_2)\, L_{a_1,N}(\lambda_1)\, L_{a_2,N}(\lambda_2) \dots L_{a_1,1}(\lambda_1)\, L_{a_2,1}(\lambda_2)\,, \end{aligned}$$

by using the ultralocality equation repeatedly. We can now apply repeatedly the intertwining equation for L-operators in order to transfer the R-operator from the left to the right of the product:

$$\begin{aligned} \dots &= L_{a_2,N}(\lambda_2)\, L_{a_1,N}(\lambda_1)\, R_{a_1,a_2}(\lambda_1,\lambda_2) \dots L_{a_1,1}(\lambda_1)\, L_{a_2,1}(\lambda_2) \\ &\qquad = L_{a_2,N}(\lambda_2)\, L_{a_1,N}(\lambda_1) \dots L_{a_2,1}(\lambda_2)\, L_{a_1,1}(\lambda_1)\, R_{a_1,a_2}(\lambda_1,\lambda_2)\,. \end{aligned}$$

Finally we use again the ultralocality equation in order to make the monodromy reappear:

$$\dots \;=\; L_{a_2,N}(\lambda_2) \dots L_{a_2,1}(\lambda_2)\, L_{a_1,N}(\lambda_1) \dots L_{a_1,1}(\lambda_1)\, R_{a_1,a_2}(\lambda_1,\lambda_2) \;=\; T_{a_2}(\lambda_2)\, T_{a_1}(\lambda_1)\, R_{a_1,a_2}(\lambda_1,\lambda_2)\,.$$

□

We define the **generating function of the conserved quantities** $Q(\lambda) \in \mathbf{E}nd(St)$ as the trace of the monodromy over the auxiliary space:

$$Q(\lambda) \stackrel{\text{def}}{=} tr_a(T_a(\lambda)) \stackrel{\text{def}}{=} \sum_{n=0}^{N} Q_n\, \lambda^n\,.$$

Corollary 2.3. *For all* $\lambda, \mu \in \mathbb{C}$ *the generating functions* $Q(\lambda)$ *and* $Q(\mu)$ *commute.*

Proof. We supposed the R-operator invertible. Hence we can rewrite the intertwining equation for the monodromy as:

$$T_{a_1}(\lambda_1)\,T_{a_2}(\lambda_2) \;=\; R^{-1}_{a_1,a_2}(\lambda_1,\lambda_2)\,T_{a_2}(\lambda_2)\,T_{a_1}(\lambda_1)\,R_{a_1,a_2}(\lambda_1,\lambda_2)\,.$$

We take the trace of this equation on $\mathbf{E}nd(A^{\otimes 2})$:

$$tr_{a_1,a_2}(T_{a_1}(\lambda)\,T_{a_2}(\mu)) \;= tr_{a_1,a_2}(R^{-1}_{a_1,a_2}(\lambda,\mu)\,T_{a_2}(\mu)\,T_{a_1}(\lambda)\,R_{a_1,a_2}(\lambda,\mu)) \;=\; tr_{a_1,a_2}(T_{a_2}(\mu)\,T_{a_1}(\lambda))$$

by using the cyclicity property of the trace on $\mathbf{E}nd(A^{\otimes 2})$. This equation reads:

$$Q(\lambda)\,Q(\mu) \;=\; Q(\mu)\,Q(\lambda)\,.$$

□

An immediate consequence of this corollary is that $\{Q_n\}_{n\in[\![1,N]\!]}$ is a family of commuting operators acting on the space of states of the system. This is a clue for integrability. Making the parallel with classical integrable systems we interpret these operators as the equivalent of the constants of the motion. Taking $Q_0 = H$ as the Hamiltonian of some system of quantum statistical mechanics, the equations of the motion for these operators are:

$$\frac{dQ_n}{dt} \stackrel{\text{def}}{=} [H, Q_n] \;=\; 0\,.$$

3. The XXX Heisenberg Spin Chain

3.1. **Definition of the System.** As a system of quantum statistical mechanics the XXX spin chain is defined by a space of states and a Hamiltonian.

The space of states is fixed by the choice $V = A = \mathbb{C}^2$ i.e. $St \;=\; \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ where we think of each copy of $\mathbb{C}^2$ as attached to one of the N sites of a one dimensional lattice. From a physical point of view the spin is related to the Lie algebra $sl_2(\mathbb{C})$. A two dimensional representation of $sl_2(\mathbb{C})$ is given by the generators:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,$$

in the canonical basis of $\mathbb{C}^2$ with the notation $|{\uparrow}\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ for spin up, $|{\downarrow}\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ for spin down. We interpret a measure on the system as applying the operators $\sigma^x, \sigma^y, \sigma^z$ on a element of $\mathbb{C}^2$.

As an example, $\sigma^z|{\uparrow}\rangle \;=\; (+1)|{\uparrow}\rangle$ and $\sigma^z|{\downarrow}\rangle \;=\; (-1)|{\downarrow}\rangle$, hence the denomination spin up for the eigenvalue $+1$ and spin down for the eigenvalue -1.

We define the operators $\sigma^\bullet_n \in \mathbf{E}nd(St)$ for $n \in [\![1, N]\!]$ and $\bullet \in \{x, y, z\}$ such that $\sigma^\bullet_n$ acts trivially everywhere except on the n^{th} site where it acts as $\sigma^\bullet$. We suppose that the spin chain is periodic that is for every $\mathcal{O}_n \in \mathbf{E}nd(St)$ whe have $\mathcal{O}_{n+N} \;=\; \mathcal{O}_n$. The Hamiltonian of the XXX spin chain $H_{xxx} \in \mathbf{E}nd(St)$ is defined by:

$$H_{xxx} \stackrel{\text{def}}{=} -\sum_{n=1}^{N} \sigma^x_n\,\sigma^x_{n+1} \;+\; \sigma^y_n\,\sigma^y_{n+1} \;+\; \sigma^z_n\,\sigma^z_{n+1}\,.$$

We can rewrite the Hamiltonian in a simpler way. For this we introduce the **permutation operator** $P_{n,m} \in \mathbf{E}nd(St)$ defined by

$$P_{n,m}\,|s_1 \dots \overset{n}{\widehat{s_n}} \dots \overset{m}{\widehat{s_m}} \dots s_N\rangle \;=\; |s_1 \dots \overset{n}{\widehat{s_m}} \dots \overset{m}{\widehat{s_n}} \dots s_N\rangle\,,$$

with the notation $|s_1 \dots s_n \dots s_m \dots s_N\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \;\dots.\; \otimes |s_N\rangle$.

Properties 3.1. *The permutation operator satisfies the following properties:*

$$(\mathbf{a}) \qquad P_{n,p}\,P_{n,q} \;=\; P_{p,q}\,P_{n,p} \;=\; P_{n,q}\,P_{p,q}\,,$$

$$(\mathbf{b}) \qquad 2\,P_{n,n+1} \;=\; \sigma^x_n\,\sigma^x_{n+1} \;+\; \sigma^y_n\,\sigma^y_{n+1} \;+\; \sigma^z_n\,\sigma^z_{n+1} \;+\; id_{n,n+1}\,.$$

Proof. We start with property $(\mathbf{a})$. We look at the action of these three operators on the same vector:

$$P_{n,p}\,P_{n,q}\,|\dots \overset{n}{\widehat{s_n}} \dots \overset{p}{\widehat{s_p}} \dots \overset{q}{\widehat{s_q}} \dots\rangle \;=\; P_{n,p}\,|\dots \overset{n}{\widehat{s_q}} \dots \overset{p}{\widehat{s_p}} \dots \overset{q}{\widehat{s_n}} \dots\rangle \;=\; |\dots \overset{n}{\widehat{s_p}} \dots \overset{p}{\widehat{s_q}} \dots \overset{q}{\widehat{s_n}} \dots\rangle\,,$$

$$P_{p,q}\,P_{n,p}\,|\dots \overset{n}{\widehat{s_n}} \dots \overset{p}{\widehat{s_p}} \dots \overset{q}{\widehat{s_q}} \dots\rangle \;=\; P_{p,q}\,|\dots \overset{n}{\widehat{s_p}} \dots \overset{p}{\widehat{s_n}} \dots \overset{q}{\widehat{s_q}} \dots\rangle \;=\; |\dots \overset{n}{\widehat{s_p}} \dots \overset{p}{\widehat{s_q}} \dots \overset{q}{\widehat{s_n}} \dots\rangle\,,$$

$$P_{n,q}\,P_{p,q}\,|\dots \overset{n}{\widehat{s_n}} \dots \overset{p}{\widehat{s_p}} \dots \overset{q}{\widehat{s_q}} \dots\rangle \;=\; P_{n,q}\,|\dots \overset{n}{\widehat{s_n}} \dots \overset{p}{\widehat{s_q}} \dots \overset{q}{\widehat{s_p}} \dots\rangle \;=\; |\dots \overset{n}{\widehat{s_p}} \dots \overset{p}{\widehat{s_q}} \dots \overset{q}{\widehat{s_n}} \dots\rangle\,,$$

which gives us the equality of these three operators.

To prove property $(\mathbf{b})$ we look at the matrices representing the operators in the canonical basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$. We use the definition of the tensor product of matrices, for $A = (a_{ij}), B = (b_{ij})$:

$$A \otimes B \;=\; \begin{pmatrix} * & * & * \\ * & a_{ij}\,B & * \\ * & * & * \end{pmatrix}\,.$$

So the matrices of these operators are:

$$P_{n,n+1} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,,\; id_{n,n+1} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,,\; \sigma^x_n\sigma^x_{n+1} = \sigma^x \otimes \sigma^x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\,,$$

$$\sigma^y_n\sigma^y_{n+1} = \sigma^y \otimes \sigma^y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\,,\; \sigma^z_n\sigma^z_{n+1} = \sigma^z \otimes \sigma^z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,.$$

Taking the sum gives us the identity. □

Usiing the property $(\mathbf{b})$ satisfied by the operator of permutation we have a simpler expression for the Hamiltonian:

$$H_{xxx} \;=\; \sum_{n=1}^{N} id_{n,n+1} \;-\; 2\,P_{n,n+1}\,.$$

The next step is to construct and diagonalise this Hamiltonian using the properties satisfied by R and L-operators.

3.2. **Algebraic Bethe Ansatz.** We need to solve the conditions for integrability with $V = A = \mathbb{C}^2$ such that we can recover the Hamiltonian for the XXX spin chain from the generating function $Q(\lambda)$.

We start with the R-operator in $\mathbf{E}nd(\mathbb{C}^{2\otimes 2})$, $\lambda_1, \lambda_2 \in \mathbb{C}$ defined by:

$$R_{a_1,a_2}(\lambda_1,\lambda_2) \overset{\text{def}}{=} f(\lambda_1,\lambda_2)\, id_{a_1,a_2} \;+\; P_{a_1,a_2}\,,$$

with $f(\lambda_1,\lambda_2) \;=\; 2\,(\lambda_2-\lambda_1)$.

Proposition 3.2. *The R-operator defined above is a solution of the Yang–Baxter equation in $\mathbf{E}nd(\mathbb{C}^{2\otimes 3})$.*

Proof. We write explicitly the left hand side and the right hand side of the Yang–Baxter equation then we identify the terms using elementary properties of the permutation operator. □

The similarity between the Yang–Baxter and the intertwining equation for L-operators is such that if we define for all $n \in [\![1, N]\!]$

$$L_{a,n}(\lambda) \overset{\text{def}}{=} (\lambda+1)\, R_{a,n}(\lambda, 0)$$

we automatically obtain a solution of the intertwining equation.

Proposition 3.3. *The L-operators defined above are solution of the intertwining equation in $\mathbf{E}nd(\mathbb{C}^{2\otimes 3})$ and satisfy the ultralocality equation in $\mathbf{E}nd(\mathbb{C}^{2\otimes N+2})$.*

Proof. We write the intertwining equation in terms of the R-operator and we use the fact that it is a solution of the Yang–Baxter equation:

$$\begin{aligned} R_{a_1,a_2}(\lambda,\mu)\, L_{a_1,n}\, L_{a_2,n} \;&=\; (\lambda+1)(\mu+1)\, R_{a_1,a_2}(\lambda,\mu)\, R_{a_1,n}(\lambda,0)\, R_{a_2,n}(\mu,0) \\ &=\; (\lambda+1)(\mu+1)\, R_{a_2,n}(\mu,0)\, R_{a_1,n}(\lambda,0)\, R_{a_1,a_2}(\lambda,\mu) \;=\; L_{a_2,n}\, L_{a_1,n}\, R_{a_1,a_2}(\lambda,\mu)\,. \end{aligned}$$

The ultralocality equation is satisfied. For $1 \le n \ne m \le N$, it is equivalent to

$$R_{a_1,n}(\lambda,0)\, R_{a_2,m}(\mu,0) \;=\; R_{a_2,m}(\mu,0)\, R_{a_1,n}(\lambda,0)$$

which is verified since the permutation operators $P_{a_1,n}$ and $P_{a_2,m}$ commute as they are acting on different spaces. □

To introduce the algebraic Bethe ansatz we use a specific decomposition of the monodromy of the system. With $A = \mathbb{C}^2$ we have $\mathbf{E}nd(A \otimes St) \cong M_2(\mathbb{C}) \otimes \mathbf{E}nd(St)$. So we can write the monodromy as:

$$T_a(\lambda) \overset{\text{def}}{=} \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{N}_+(\lambda) \\ \mathcal{N}_-(\lambda) & \mathcal{D}(\lambda) \end{pmatrix},$$

with $\mathcal{A}(\lambda), \mathcal{N}_+(\lambda), \mathcal{N}_-(\lambda), \mathcal{D}(\lambda) \in \mathbf{E}nd(St)$ and $Q(\lambda) = \mathcal{A}(\lambda) + \mathcal{D}(\lambda)$. In order to prove the algebraic Bethe ansatz theorem we need a technical lemma.

Lemma 3.4. *The operators $\mathcal{A}, \mathcal{N}_+, \mathcal{D}$ satisfy the following relations for $g(\lambda,\mu) = 1 + f(\lambda,\mu)$:*

$$\mathcal{N}_+(\lambda)\,\mathcal{N}_+(\mu) \;=\; \mathcal{N}_+(\mu)\,\mathcal{N}_+(\lambda)\,,$$

$$\mathcal{A}(\lambda)\,\mathcal{N}_+(\mu) \;=\; \frac{g(\mu,\lambda)}{f(\mu,\lambda)}\,\mathcal{N}_+(\mu)\,\mathcal{A}(\lambda) \;-\; \frac{1}{f(\mu,\lambda)}\,\mathcal{N}_+(\lambda)\,\mathcal{A}(\mu)\,,$$

$$\mathcal{D}(\lambda)\,\mathcal{N}_+(\mu) \;=\; \frac{g(\lambda,\mu)}{f(\lambda,\mu)}\,\mathcal{N}_+(\mu)\,\mathcal{D}(\lambda) \;-\; \frac{1}{f(\lambda,\mu)}\,\mathcal{N}_+(\lambda)\,\mathcal{D}(\mu)\,.$$

Proof. We extract these relations from the intertwining equation for the monodromy in $\mathbf{E}nd(A\otimes A\otimes St) \cong M_4(\mathbb{C})\otimes \mathbf{E}nd(St)$:

$$R_{a_1,a_2}(\lambda,\mu)\,T_{a_1}(\lambda)\,T_{a_2}(\mu) \;=\; T_{a_2}(\mu)\,T_{a_1}(\lambda)\,R_{a_1,a_2}(\lambda,\mu)\,.$$

We write $R_{a_1,a_2}(\lambda,\mu), T_{a_1}(\lambda), T_{a_2}(\mu)$ as 4 by 4 matrices whose coefficients are operators on the space of states of the system such that the intertwining equation corresponds to a product of matrices. We work in the canonical basis of $\mathbb{C}^2\otimes\mathbb{C}^2$.

The R-operator acts only on $A\otimes A$:

$$R_{a_1,a_2}(\lambda,\mu) \;=\; f(\lambda,\mu)\,id_{a_1,a_2} \;+\; P_{a_1,a_2} \;=\; \begin{pmatrix} g(\lambda,\mu) & 0 & 0 & 0\\ 0 & f(\lambda,\mu) & 1 & 0\\ 0 & 1 & f(\lambda,\mu) & 0\\ 0 & 0 & 0 & g(\lambda,\mu)\end{pmatrix}.$$

The operator $T_{a_1}(\lambda)$ acts trivially on the second copy of $A\otimes A$ and $T_{a_2}(\mu)$ acts trivially on the first copy of $A\otimes A$, so we get:

$$T_{a_1}(\lambda) \;=\; \begin{pmatrix}\mathcal{A}(\lambda) & \mathcal{N}_+(\lambda)\\ \mathcal{N}_-(\lambda) & \mathcal{D}(\lambda)\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \;=\; \begin{pmatrix}\mathcal{A}(\lambda) & 0 & \mathcal{N}_+(\lambda) & 0\\ 0 & \mathcal{A}(\lambda) & 0 & \mathcal{N}_+(\lambda)\\ \mathcal{N}_-(\lambda) & 0 & \mathcal{D}(\lambda) & 0\\ 0 & \mathcal{N}_-(\lambda) & 0 & \mathcal{D}(\lambda)\end{pmatrix},$$

$$T_{a_2}(\mu) \;=\; \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\begin{pmatrix}\mathcal{A}(\mu) & \mathcal{N}_+(\mu)\\ \mathcal{N}_-(\mu) & \mathcal{D}(\mu)\end{pmatrix} \;=\; \begin{pmatrix}\mathcal{A}(\mu) & \mathcal{N}_+(\mu) & 0 & 0\\ \mathcal{N}_-(\mu) & \mathcal{D}(\mu) & 0 & 0\\ 0 & 0 & \mathcal{A}(\mu) & \mathcal{N}_+(\mu)\\ 0 & 0 & \mathcal{N}_-(\mu) & \mathcal{D}(\mu)\end{pmatrix}.$$

We compute the product of these matrices and keep only the terms corresponding to our relations:

$$\begin{pmatrix} * & * & g(\lambda,\mu)\,\mathcal{N}_+(\lambda)\,\mathcal{A}(\mu) & g(\lambda,\mu)\,\mathcal{N}_+(\lambda)\,\mathcal{N}_+(\mu) \\ * & * & * & * \\ * & * & * & \mathcal{N}_+(\lambda)\,\mathcal{D}(\mu) + f(\lambda,\mu)\,\mathcal{D}(\lambda)\,\mathcal{N}_+(\mu) \\ * & * & * & * \end{pmatrix}$$
$$= \begin{pmatrix} * & * & \mathcal{N}_+(\mu)\,\mathcal{A}(\lambda) + f(\lambda,\mu)\,\mathcal{A}(\mu)\,\mathcal{N}_+(\lambda) & g(\lambda,\mu)\,\mathcal{N}_+(\mu)\,\mathcal{N}_+(\lambda) \\ * & * & * & * \\ * & * & * & g(\lambda,\mu)\,\mathcal{N}_+(\mu)\,\mathcal{D}(\lambda) \\ * & * & * & * \end{pmatrix} .$$

The identification of the coefficients gives the relations. □

We have all that is necessary to introduce the algebraic Bethe ansatz. A **pseudo-vacuum** of the system is a vector $|\omega\rangle \in St$ such that:

$$\mathcal{N}_-(\lambda)|\omega\rangle \;=\; 0\,,\; \mathcal{A}(\lambda)|\omega\rangle \;=\; \alpha(\lambda)|\omega\rangle\,,\; \mathcal{D}(\lambda)|\omega\rangle \;=\; \delta(\lambda)|\omega\rangle\,.$$

Theorem 3.5 (Algebraic Bethe Ansatz). *The vector $\mathcal{N}_+(\mu_1)\ldots\mathcal{N}_+(\mu_M)|\omega\rangle$ is an eigenvector of $Q(\lambda)$ with the eigenvalue*

$$\alpha(\lambda)\prod_{k=1}^{M}\frac{g(\mu_k,\lambda)}{f(\mu_k,\lambda)} \;+\; \delta(\lambda)\prod_{k=1}^{M}\frac{g(\lambda,\mu_k)}{f(\lambda,\mu_k)}$$

if and only if for all $i \in [\![1,M]\!]$, μ_i is solution of Bethe equation

$$\frac{\alpha(\mu_i)}{\delta(\mu_i)}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{g(\mu_j,\mu_i)}{g(\mu_i,\mu_j)} \;=\; (-1)^{M-1}\,.$$

Proof. Since $Q(\lambda) = \mathcal{A}(\lambda) + \mathcal{D}(\lambda)$ we can use the lemma to see how $\mathcal{A}$ and $\mathcal{D}$ are acting on the possible eigenvector:

$$\mathcal{A}(\lambda)\prod_{k=1}^{M}\mathcal{N}_+(\mu_k)|\omega\rangle \;=\; \alpha(\lambda)\prod_{k=1}^{M}\frac{g(\mu_k,\lambda)}{f(\mu_k,\lambda)}\,\prod_{l=1}^{M}\mathcal{N}_+(\mu_l)|\omega\rangle$$
$$+\;\sum_{i=1}^{M} -\frac{\alpha(\mu_i)}{f(\mu_i,\lambda)}\,\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{g(\mu_j,\mu_i)}{f(\mu_j,\mu_i)}\,\mathcal{N}_+(\lambda)\prod_{\substack{k=1\\ k\neq i}}^{M}\mathcal{N}_+(\mu_k)|\omega\rangle\,,$$

$$\mathcal{D}(\lambda)\prod_{k=1}^{M}\mathcal{N}_+(\mu_k)|\omega\rangle \;=\; \delta(\lambda)\prod_{k=1}^{M}\frac{g(\lambda,\mu_k)}{f(\lambda,\mu_k)}\prod_{l=1}^{M}\mathcal{N}_+(\mu_l)|\omega\rangle$$
$$+\sum_{i=1}^{M}-\frac{\delta(\mu_i)}{f(\lambda,\mu_i)}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{g(\mu_i,\mu_j)}{f(\mu_i,\mu_j)}\mathcal{N}_+(\lambda)\prod_{\substack{k=1\\ k\neq i}}^{M}\mathcal{N}_+(\mu_k)|\omega\rangle\,.$$

We observe that in order for $\mathcal{N}_+(\mu_1)\,...\mathcal{N}_+(\mu_M)|\omega\rangle$ to be an eigenvector we need that for all $i\in[\![1,M]\!]$:

$$\frac{\alpha(\mu_i)}{f(\mu_i,\lambda)}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{g(\mu_j,\mu_i)}{f(\mu_j,\mu_i)} + \frac{\delta(\mu_i)}{f(\lambda,\mu_i)}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{g(\mu_i,\mu_j)}{f(\mu_i,\mu_j)} \;=\; 0\,.$$

But since $f(\lambda,\mu)=2\,(\mu-\lambda)=-f(\mu,\lambda)$, dividing by the second term of the sum gives Bethe equation. □

Explicitly for the XXX spin chain, the monodromy of the system is the product of the R-operators:

$$T_a(\lambda) \;=\; (\lambda+1)^N\,R_{a,N}(\lambda,0)\,...\,R_{a,1}(\lambda,0)\,.$$

We write the R-operator as a 2 by 2 matrix with coefficients $\widehat{a}_n,\widehat{d}_n,\widehat{\eta}_{+,n},\widehat{\eta}_{-,n}\in\mathbf{E}nd(V)$ acting on the n^{th} local space of states:

$$R_{a,n}(\lambda,0) \;=\; \begin{pmatrix}\widehat{a}_n(\lambda) & \widehat{\eta}_{+,n}(\lambda)\\ \widehat{\eta}_{-,n}(\lambda) & \widehat{d}_n(\lambda)\end{pmatrix}\,,$$

with

$$\widehat{a}_n(\lambda) \;=\; \begin{pmatrix}g(\lambda,0) & 0\\ 0 & f(\lambda,0)\end{pmatrix},\widehat{d}_n(\lambda) \;=\; \begin{pmatrix}f(\lambda,0) & 0\\ 0 & g(\lambda,0)\end{pmatrix},\widehat{\eta}_{+,n} \;=\; \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\widehat{\eta}_{-,n} \;=\; \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

From the form of $R_{a,n}(\lambda,0)$ we see that $|\omega\rangle\;=\;|\!\uparrow\rangle\otimes|\!\uparrow\rangle\otimes\,...\,\otimes|\!\uparrow\rangle$ is a pseudo-vacuum of the system with $\alpha(\lambda)\;=\;(\lambda+1)^N\,g(\lambda,0)^N$ and $\delta(\lambda)\;=\;(\lambda+1)^N\,f(\lambda,0)^N$. The eigenvalues of $Q(\lambda)$ are

$$[(\lambda+1)(1-2\lambda)]^N\prod_{k=1}^{M}\frac{1+2(\mu_k-\lambda)}{2(\mu_k-\lambda)} + [(\lambda+1)(-2\lambda)]^N\prod_{k=1}^{M}\frac{1+2(\lambda-\mu_k)}{2(\lambda-\mu_k)}$$

with the μ_i solutions of Bethe equation

$$\frac{2\mu_i-1}{2\mu_i}\prod_{\substack{k=1\\ k\neq i}}^{M}\frac{1+2(\mu_k-\mu_i)}{1+2(\mu_i-\mu_k)} \;=\; (-1)^{M-1}\,.$$

Finally we should not forget to prove that this formalism solves the problem of the XXX spin chain i.e. that we can recover the Hamiltonian of the system from the generating function $Q(\lambda)$.

Proposition 3.6. *The Hamiltonian of the system is given by*

$$H_{xxx} = \left.\frac{d\,lnQ(\lambda)}{d\lambda}\right|_{\lambda=0} .$$

Proof. First of all we need to compute $Q(0)$ and its inverse. Since $R_{a,n}(0,0) = P_{a,n}$:

$$Q(0) = tr_a(R_{a,N}(0,0)\dots R_{1,a}(0,0)) = tr_a(P_{a,N}\,P_{a,N-1}\dots P_{a,1})\,.$$

By using the property (**a**) of the permutation operator we obtain:

$$Q(0) = tr_a(P_{N,N-1}\,P_{a,N}\,P_{a,N-2}\dots P_{a,1}) = \dots = P_{N,N-1}\,P_{N,N-2}\dots P_{N,1}\,tr_a(P_{N,a})$$

and

$$Q(0)^{-1} = \frac{1}{tr_a(P_{N,a})}\,P_{N,1}\dots P_{N,N-2}\,P_{N,N-1}\,.$$

We also need to compute $dQ/d\lambda$:

$$\frac{dQ}{d\lambda} = N(\lambda+1)^N\,tr_a(R_{a,N}(\lambda,0)\dots R_{a,1}(\lambda,0)) + (\lambda+1)^N\,tr_a\left(\sum_{n=1}^{N} R_{a,N}(\lambda,0)\dots\frac{dR_{a,n}(\lambda,0)}{d\lambda}\dots R_{a,1}(\lambda,0)\right)$$

with $dR_{a,n}/d\lambda = -2\,id_{a,n}$. Using again property (**a**) we end up with:

$$\left.\frac{d\,lnQ(\lambda)}{d\lambda}\right|_{\lambda=0} = N\,id_{St} - 2\sum_{n=1}^{N} Q(0)^{-1}\,P_{N,N-1}\dots P_{N,n+1}\,P_{N,n-1}\dots P_{n,1}\,tr_a(P_{a,N}) = N\,id_{St} - 2\sum_{n=1}^{N} P_{n,n-1}\,.$$

Because of the periodicity of the chain we conclude that it is the Hamiltonian of the system:

$$\left.\frac{d\,lnQ(\lambda)}{d\lambda}\right|_{\lambda=0} = \sum_{n=1}^{N} id_{n,n+1} - 2P_{n,n+1}\,.$$

□

4. The XXZ Heisenberg Spin Chain

5. The Six–Vertex Model and Alternating Sign Matrices

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