Combinatorial Examples Session 1

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Today's lecture is based upon Section 3 of [1].

We let $\mathcal{C}$ be a combinatorial class with $d-1$ parameters being counted and $\mathbf{r} \in \mathbb{N}^d$ be a d-dimensional multi-index. We then have the multivariate generating function for $\mathcal{C}$

$$F(z) = a_{\mathbf{r}} z^{\mathbf{r}}.$$

which for simplification purposes is always of the form

$$F(z) = \frac{G(z)}{H(z)},$$

G, H analytic, $H(\mathbf{0}) \neq 0$. We are then interested in finding an asymptotic formula for $a_{\mathbf{r}}$ of the form

$$a_{\mathbf{r}} \sim \sum_{z \in \beta contrib_{\bar{\mathbf{r}}}} \mathbf{formula}(\mathbf{z}). \tag{1}$$

as $|\mathbf{r}| \to \infty$ and $\beta contrib_{\bar{\mathbf{r}}}$ is a set of contributing critical points dependent on the direction of $\mathbf{r}$, denoted $\bar{\mathbf{r}}$.

Indeed, it is useful to separate $\mathbf{r}$ into a scale parameter $|\mathbf{r}|$, a positive real number, and a direction $\bar{\mathbf{r}}$, which is an element of real projective space. While it is true that $\mathbf{r} \in \mathbb{R}^d$, it is sometimes useful to consider it in $\mathbb{C}^d$ or as an element of real or complex projective space. In the source [1], it is noted that $\mathbf{r}$ could be anywhere in the following diagram, where $\hat{\mathbf{r}} \in \Delta^{d-1}$ is the projection of $\bar{\mathbf{r}}$ in the real $d-1$ simplex.

$$\begin{array}{ccccc}
\mathcal{O} & \rightarrow & \mathbb{R}^d & \rightarrow & \mathbb{C}^d \\
{}^{-}\downarrow & & {}^{-}\downarrow & & {}^{-}\downarrow \\
\bar{\mathcal{O}} & \rightarrow & \mathbb{RP}^{d-1} & \rightarrow & \mathbb{CP}^{d-1} \\
{}^{\wedge}\downarrow & & & & \\
\Delta^{d-1} & & & &
\end{array}$$

Suppose that F is a rational function for simplicity. Then let $\mathcal{V} = \{\mathbf{z} : H(\mathbf{z}) = 0\}$ be the pole variety of F. Then the process of finding the asymptotics is as follows.

1. Asymptotics in the direction $\bar{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a finite set $\beta crit_{\bar{\mathbf{r}}}$ of *critical points*.
2. We then reduce this set further to $\beta contrib_{\bar{\mathbf{r}}} \subseteq \beta crit_{\bar{\mathbf{r}}}$ of *contributing critical points*, usually a single point.
3. We determine both of the above sets by a combination of algebraic and geometric criteria.
4. Critical points are either smooth, multiple or bad.
5. Corresponding to each smooth or multiple point, $\mathbf{z}$, is an asymptotic expression for $a_{\mathbf{r}}$ which is computable in terms of G and H at $\mathbf{z}$.

These steps culminate in the meta-formula (1), which is different for multiple points and smooth points. There is no formula for bad points.

Central to the derivation is the Cauchy Integral Formula

$$a_{\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) d\mathbf{z},$$

where T is a torus about the origin, in which each circle is of sufficiently small radius in each coordinate. The main idea is to replace the torus T by a product $T_1 \times T_2$, where the inner integral over T_1 is a multivariate residue, and the outer integral over T_2 is a saddle point bound. This leads to a complete asymptotic series which may be read off in a straightforward manner. We don't do this in this document, but instead show an application of the results to the class of binomial coefficients, which are well known and will provide a hook for readers.

Before we do move on, a brief note about the method. The main tool that is used is a Whitney stratification of the pole variety $\mathcal{V}$. For rational F, this will give quickly the set $\beta crit_{\bar{\mathbf{r}}}$ as a finite union of zero dimensional varieties and elimination theory gives minimal polynomials in an automatic way. The CIF is then reduced to a sum of integrals over C_i:

$$\sum_{n_i} \int_{C_i} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) d\mathbf{z};$$

where the C_i are quasi-local cycles near $\mathbf{z}$ and $T = \sum n_i C_i$ (we think of n_i like a winding number). By considering the height function associated with the stratification, we are then able to reduce to the smaller set $\beta contrib_{\bar{\mathbf{r}}} \subseteq \beta crit_{\bar{\mathbf{r}}}$. From here geometry is required and the answer will differ based on whether the critical points, $\bar{\mathbf{z}}$, are smooth ($\mathcal{V}$ is locally a manifold) or multiple ($\mathcal{V}$ is locally the union of finitely many manifolds intersecting transversally). Finally, we define a linear space $L(\mathbf{z})$.

Definition 1 *Let S be a strata of $\mathcal{V}$ and $\mathbf{z} \in S$. Then $L(\mathbf{z}) \in \mathbb{CP}^{d-1}$ is the span of the projections of vectors orthogonal to the tangent space of S at $\mathbf{z}$.*

All of this culminates, for us, in the following theorem, where $\mathbf{z}(\bar{\mathbf{r}})$ is the solution $\mathbf{z}$ to the 'equation' $\bar{\mathbf{r}} \in L(\mathbf{z})$.

Theorem 1 *When $d = 2$ and $G(\mathbf{z}(\bar{\mathbf{r}})) \neq 0$,*

$$a_{r,s} \sim \frac{G(x,y)}{\sqrt{2\pi}} x^{-r} y^{-s} \sqrt{\frac{yH_y}{sQ(x,y)}},$$

where $(x,y) = \mathbf{z}(\bar{(r,s)})$ and

$$Q(x,y) = -(xH_x)(yH_y)^2 - (yH_y)(xH_x)^2 - \left[(yH_y)^2 x^2 H_{xy} + (xH_x)^2 y^2 H_{yy} - 2(xH_x)(yH_y)xyH_{xy}\right].$$

So, take $\mathcal{B}$ to be the class of binomial coefficients. This is a well known class, and has generating function

$$F(x,y) = \frac{1}{1-x-y},$$

with $a_{r,s} = \binom{r+s}{r}$.

The pole variety is the complex line $\mathcal{V} = \{(x,y) | x+y = 1\}$. The numerator never vanishes, so we may apply the above theorem. For any $\mathbf{z}$, $L(\mathbf{z})$ is the linear span of $\mathbf{z}$ [1, Prop 3.11]. Thus, for each $\bar{\mathbf{r}}$ in the positive real orthant, we are able to find a unique $\mathbf{z} \in \mathcal{V}$ solving $\bar{\mathbf{r}} \in L(\mathbf{z})$, namely $\mathbf{z} = \left(\frac{r}{r+s}, \frac{s}{r+s}\right)$. An application of the machinery that we outlined above shows that $\beta contrib_{\bar{\mathbf{r}}} = \left\{\left(\frac{r}{r+s}, \frac{s}{r+s}\right)\right\}$.

We need only compute $Q(x, y)$ and we're done. We have $H = 1 - x - y$, $H_x = -1 = H_y$ and all other partial derivatives zero, so $Q(x, y) = -xy(x + y)$. Substituting all this into the above theorem gives

$$a_{r,s} \sim \left(\frac{r+s}{r}\right)^r \left(\frac{r+s}{s}\right)^s \sqrt{\frac{r+s}{2\pi rs}}.$$

This asymptotic expressionis valid as $(r, s) \to \infty$ uniformly if r/s and s/r remain bounded.

References

[1] R Pemantle and M Wilson. Twenty combinatorial examples of asymptotics derived from multivariate generating functions. *arXiv:math/0512548v2*, 2007.