ASSIGNMENT 4 SOLUTIONS

MATH 303, FALL 2011

If you find any errors please let me know.

Manipulation

(M1) No the integers with the usual $\leq$ are not well ordered. We can see this because the set of all integers has no least element, since for any $n \in \mathbb{Z}$, $n-1 \in \mathbb{Z}$ and $n-1 < n$. So it is certainly not the case that all subsets of the set of all integers have a least element, and hence this is not a well order.

(M2) $A \times B = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$, which has size 6. *Notice that* $6 = \#(A)\#(B)$.

(M3)

$$\begin{aligned} B^A = \{&\{(8,6),(96,6)\},\{(8,6),(96,7)\},\{(8,6),(96,8)\},\\ &\{(8,7),(96,6)\},\{(8,7),(96,7)\},\{(8,7),(96,8)\},\\ &\{(8,8),(96,6)\},\{(8,8),(96,7)\},\{(8,8),(96,8)\}\} \end{aligned}$$

which has size 9. *Notice that* $9 = (\#(B))^{\#(A)}$.

(M4)

$$\begin{aligned} \mathcal{P}(A) = \{&\emptyset,\{5\},\{7\},\{83\},\{251\}\\ &\{5,7\},\{5,83\},\{5,251\},\{7,83\},\{7,251\},\{83,251\},\\ &\{5,7,83\},\{5,7,251\},\{5,83,251\},\{7,83,251\},\\ &\{5,7,83,251\}\} \end{aligned}$$

which has size 16. *Notice that* $16 = 2^{\#(A)}$.

Pure Math

(P1) Define $f : (Z^Y)^X \to Z^{X\times Y}$ as follows. For every $g \in (Z^Y)^X$, $f(g)$ is the function in $Z^{X\times Y}$ which takes $(x,y) \in X \times Y$ to $(g(x))(y)$. Now we just need to check f is one-to-one and onto.

one-to-one: Suppose $g_1, g_2 \in (Z^Y)^X$ and $f(g_1) = f(g_2)$ then for every $(x,y) \in X \times Y$,

$$(g_1(x))(y) = f(g_1)(x,y) = f(g_2)(x,y) = (g_2(x))(y)$$

Thus for every fixed $x \in X$ we have that $g_1(x)$ and $g_2(x)$ agree on all elements of their domain (namely Y), and so they are the same function. Thus g_1 and g_2 agree on all elements of their domain (namely X) and so they are the same function, that is $g_1 = g_2$.

onto: Take $h \in Z^{X\times Y}$. Define $g \in (Z^Y)^X$ by saying that $g(x)$ is the function which takes $y \in Y$ to $h(x,y)$. Then $f(g)$ is the function which takes $(x,y) \in X \times Y$ to $(g(x))(y) = h(x,y)$, that is $f(g) = h$.

Did anyone with a CS background notice that P1 was about currying?

(P2) Take any $A, B, C \in \mathcal{P}(E)$.

We know from class that $A \subseteq A$. This is the first thing we need to check for a partial order.

If $A \subseteq B$ and $B \subseteq A$ then this tells us that all elements of A are elements of B and all elements of B are elements of A; thus A and B have the same elements, so $A = B$. This is the second thin we need to check for a partial order.

If $A \subseteq B$ and $B \subseteq C$ then all elements of A are elements of B and all elements of B are elements of C and so in particular all elements of A are elements of C; thus $A \subseteq C$. This is the third thing we need to check for a partial order.

Thus $\mathcal{P}(E)$ with the subset relation is a partially ordered set.

This partial order is not usually a total order. For example if $E = \{a, b\}$, then $\{a\}$ and $\{b\}$ are both in $\mathcal{P}(E)$, but $\{a\} \not\subseteq \{b\}$ and $\{b\} \not\subseteq \{a\}$, so we do not have a total order.

(P3)
(a) Let $I = \omega$. This is the index set of our family. The family itself is the function $f : I \to \{0, 1\}$ given by $f(i) = a_i$ for all $i \in \omega$. In particular then the codomain is $\{0, 1\}$.
(b) If the question was reposed with $a_i \in A$, then the only thing that would change is the codomain, which would become A.

Ideas

(I1) The idea here is that sets of functions behave like exponentiations of integers and cartesian products behave like products of integers in a few ways. First of all from M2 and M3 we can guess that the sizes of the sets behave as follows

$$(\#(Y))^{\#(X)} = \#(Y^X) \quad \text{and} \quad (\#(X))(\#(Y)) = \#(X \times Y)$$

and in fact this is true in general (you could prove that for this question, but you don't have to).

Furthermore, from P1 we can guess that the identities for exponentiation and multiplication of integers give one-to-one and onto maps between the corresponding sets; at least we know that

$$(Z^Y)^X \quad \text{and} \quad Z^{X \times Y}$$

are in one-to-one correspondence, in analogy with the exponentiation rule

$$(a^b)^c = a^{bc}$$

You could continue this question by considering other rules for exponentiation and multiplication of integers and seeing if they also carry over to sets. Can you find other connections?

(I2) *answers will vary*