Math 303, Fall 2011, Lecture 17

① A brief introduction to model theory

(Following Cohen ch I section 4)

To use logic to talk about set theory we only needed one special symbol (as opposed to the general logical symbols)

Namely

is a

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—

If we want to talk about other areas of mathematics on their own terms (not as built of set theory) then we need other relations

eg

eg

eg we can make any function into a relation in the same way eg We can also write relations like $\leq$ or $\in$ in this format

say $R_{\leq}(a,b)$ means

$R_{\in}(a,b)$ means

so $R_{\leq}(1,2)$ is

but $R_{\leq}(2,1)$ is

We can expand our well formed formulas to include such relations by

Now say we have a set S of sentences in the formal language involving constant symbols

$$c_i \quad \text{for } i \in I \qquad \text{with } I \text{ some index set}$$

and relations

$$R_j \quad \text{for } j \in J \qquad \text{with } J \text{ some index set.}$$

We may assume all the sentences in S are good.

eg

$$S = \{ \forall x \forall y \forall z \left(R_+(x,y,z) \longleftrightarrow R_+(y,x,z) \right),$$
$$\forall x \forall z \left(R_+(x,0,z) \longleftrightarrow z = x \right) \}$$

This is a set of sentences involving

in usual notation they say

for the first:

for the second:

What we have so far: $c_i \quad i \in I$

$R_j \quad j \in I$

S

Next say we have a set M

and maps $f(c_i) = \overline{c}_i \in M$

$$g(R_j) = \overline{R}_j \subseteq \underbrace{M \times \ldots \times M}_{n \text{ times}}$$

where R_j takes n inputs.

These give us interpretations for the constant symbols and relation symbols within M.

eg let S, c_i, R_j be as in the previous example
let $M = \mathbb{Z}$ the set of integers

then we can let

$$f(0) =$$

$$g(R_+) =$$

then

With this interpretation

eg We could also make a silly interpretation

eg $f(0) =$

$g(R_+) =$

lets check if the sentences of S are true in this interpretation.

$$\forall x \forall y \forall z \left(R_+(x,y,z) \longleftrightarrow R_+(y,x,z) \right)$$

$$\forall x \forall z \left(R_+(x,0,z) \longleftrightarrow z=x \right) \}$$

For the break

try this interpretation

$$M = \mathbb{Z}$$

$$f(0) = 1$$

$$g(R_+) = \{ (a,b,c) : a,b,c \in \mathbb{Z},\ ab=c \}$$

Are the sentences of S true in this interpretation?

lets check

So far when checking if sentences are true in a given interpretation we have relied on our intuitive understanding of truth.

In practice this is often the best way, but we don't need to rely on it

Definition Let A be a good formula with free variables among $x_1, \dots, x_n$. Let $\bar{x}_1, \dots, \bar{x}_n$ be elements of M. The **truth of A evaluated at $\bar{x}_1, \dots, \bar{x}_n$** is defined recursively as follows

① If A is of the form

$$x_i = x_j \qquad \text{or} \qquad x_i = c_j \qquad \text{or} \qquad c_i = c_j$$

Then A is true at $\bar{x}_1, \dots, \bar{x}_n$ if

, ,

respectively, in M

(2) If A is $R(t_1, .., t_m)$ where R is a relation symbol with m inputs and each t_i is a constant symbol or one of the x_j, then A is true at $x_1, .., x_n$

if

(3) If A is a propositional function of formulas, then

(4) If A is of the form

$\forall y\, B(y, x_1, .., x_n)$ or $\exists y\, B(y, x_1, .., x_n)$

then A is true at $x_1, .., x_n$ if

,

respectively.

eg $\forall x \forall z \left(R_+(x,0,z) \longleftrightarrow z = x \right)$ in $\mathbb{Z}$ with the usual (first) interpretation

is true if

The point of all this is the following definition

Definition If S is a set of sentences involving

- constant symbols c_i, $i \in I$
- relation symbols R_j, $j \in J$

And M is a set with an interpretation given by

- $f(c_i) = \overline{c}_i \in M$
- $g(R_j) = \overline{R}_j \in \underbrace{M \times \ldots \times M}_{n \text{ times}}$ where R_j takes n inputs

Then we say M (with the interpretation) is a **model** for S if all sentences of S are true in M

eg If $S = \{ \forall x \forall y \forall z (R_+(x,y,z) \longleftrightarrow R_+(y,x,z)),$

$\forall x \forall z (R_+(x,0,z) \longleftrightarrow z = x) \}$

Then

Trying to understand how the set of sentences influences the possible models is called model theory.

In this class we are mostly interested in

We have an intuitive model in mind

but

② Next time

- Important results of model theory
- Back to Halmos to review partial orders and well orders

Please read Cohen ch 1, sections 4 and 5 (don't worry about the proofs)
And Halmos chapter 14