Math 303, Fall 2011, Lecture 19

1 Review of well orders

Recall that if X with $\leq$ is a partially ordered set and

(4)

(5)

then we say $\leq$ is a **well order** and we say X is **well ordered** by $\leq$

Note

Note

total order

We saw that ω is well ordered by the usual $\leq$

eg Consider $\omega \times \omega$

What ordering shall we use?

(a)

eg

Is $\omega \times \omega$ well ordered?

(b)

lexicographic order

Is $\omega \times \omega$ well ordered?

② Transfinite induction

Well ordered sets have the nice property that we have a notion of induction. First we need the following definition

Definition Let X with $\leq$ be a partially ordered set. Take $a \in S$.

initial segment

Let X be a well ordered set and let $S \subseteq X$

if

Then $S = X$

This is called the principle of transfinite induction

First lets see that this is true and then see what we can do with it

To check the fact suppose S has the property but $S \neq X$.

How does this relate to the principle of mathematical induction which we have already seen?

let $X = \omega$

First

Take $0 \in \omega$

$s(0) =$

Next note

strong induction

weak induction

For ω

But for other well ordered sets transfinite induction is necessary

eg let $X = \omega^+ = \omega \cup \{\omega\}$

use

Suppose we try to use the old principle of mathematical induction on X. What goes wrong?

For the break

$\boxed{\text{Can you find an } S \subsetneq \omega^+ \text{ with } 0 \in S \text{ and for all } n \in S,\ n^+ \in S\text{?}}$

answer

Transfinite induction fixes this problem

3 Ordinals

We had

Now consider

Suppose f is a function with domain $n \in \omega$

Say f is an ω-successor function

if $f(0) = \omega$

and $f(m^+) = (f(m))^+$

eg $n = 3 = \{0, 1, 2\}$.

$f(0) =$

$f(1) =$

$f(2) =$

In fact for each n there is a unique ω-successor function

Suppose f and g were both ω-successor functions with domain n

-
- Let $i \in n$ be the smallest number for which $f(i) \neq g(i)$.
-

Thus f is unique.

What we want is to join all those things togethe

Let $S(n, x)$ be the property

"$n \in \omega$ and x is in the range of an ω-successor function with domain n"

in logic

The set we are looking for is

$\{ \qquad \}$

We only know

Intuitively

We want to know

either (a)

or (b)

These are equivalent.

(b) would come from Cohen's version of the axiom of replacement

(a) is Halmos' version which he calls the axiom of substitution

Axiom of Substitution If $S(a,b)$ is a sentence such that for each $a \in A$ the set $\{b : S(a,b)\}$ can be formed, then there exists a function F with domain A such that $F(a) = \{b : S(a,b)\}$ for each $a \in A$

Our use of the axiom of substitution will be, as above, to extend our ability to count beyond $\omega, \omega^+, \ldots$

So we have

$$0, 1, 2, 3, \ldots\ldots$$

$$\omega, \omega^+, (\omega^+)^+, ((\omega^+)^+)^+, \ldots \qquad (*)$$

and by the above we can define a set theoretic function F with domain ω such that

$$F(0) = \omega, \quad F(n^+) = (F(n))^+$$

let X be the range of F

Then the next number after the ones in $(*)$ is

$$X \cup \omega = \{0, 1, 2, \ldots, \omega, \omega^+, (\omega^+)^+, \ldots\}$$

Note

What are these new bigger counting "numbers"

ordinals

Definition An ordinal is a well ordered set S such that for all $x \in S$ $s(x) = x$

eg lets check 3 is an ordinal

Likewise every natural number is an ordinal.

eg check ω is an ordinal.

Two useful facts

① If X is an ordinal then X^+ is an ordinal

proof Use the order on X^+ given by

This is a well order as

Finally we can check the ordinal property.

(2) Let X be a set. There is at most one well order which makes X into an ordinal

proof Suppose there is a well order which makes X into an ordinal. Take any other well order of X

(4) Next time

More on ordinals

please read Halmos section 20