Math 303, Fall 2011, Lecture 20

① Properties of ordinals and well orders

Every element of an ordinal number X is also a subset of X
ie if X is an ordinal and $x \in X$ then $x \subseteq X$

proof Let X be an ordinal and take $x \in X$

Note this is a property we already saw for natural numbers

Next lets define two well ordered sets X and Y to be similar if there is a function $f: X \to Y$ which is

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- for all $x_1, x_2 \in X$

eg let $X = \{2, 8, 9\}$ with $2 \leq 8 \leq 9$

and let $Y = 3$

then

but

eg lets order ω^+ in 2 ways

$X = \omega^+$ with the usual ordering ($n \leq \omega$ for all $n \in \omega$ and natural numbers are ordered as in ω)

$Y = \omega^+$ ordered by

$$\omega \leq 0 \leq 1 \leq 2 \leq \ldots$$

Is X similar to Y?

Is Y similar to something we've seen before?

More facts

If two well ordered sets are similar then there is exactly one similarity function between them

proof Let X and Y be the sets.

Say $f: X \to Y$ and $g: X \to Y$ are both similarities

Now consider the set

$$\{x \in X : h(x) < x\}$$

X is well ordered so this set has a least element call it a.

A well ordered set cannot be similar to any of its initial segments

proof Let X be a well ordered set and $x \in X$ so that X is similar to $s(x)$

Say with $f : X \to s(x)$

As for the previous fact consider the set $\{y \in X : f(y) < y\}$

Let X and Y be well ordered sets. Either X and Y are similar or one of them is similar to an initial segment of the other

proof Let $S = \{ a \in X : \exists b \in Y \, (s(a) \text{ is similar to } s(b)) \}$

Let $T = \{ b \in Y : \exists a \in S \, (s(a) \text{ is similar to } s(b)) \}$

Note

Now either $S = X$ or $X - S \neq \emptyset$ and so $X - S$ has a least element x.

Claim $S = s(x)$

(2) Next time

- Bringing the above back to ordinals
- Sizes of sets
  - Schröder - Bernstein theorem

Please read Halmos sections 22 and 23