Math 303, Fall 2011, Lecture 8

① Our axioms so far

from lecture 1

Axiom of extension two sets are equal if and only if they have the same elements

Axiom of Specification or Subset Selection
For every set A and every condition $S(x)$ there is a set B consisting of exactly the elements of A for which $S(x)$ holds

ie $$B = \{ x \in A \mid S(x) \text{ is true} \}$$

from lecture 2

Axiom of the empty set There is a set, written $\emptyset$, which contains no elements

Axiom of pairing (or unordered pairs)

For any two sets A and B, there is a set C with $A \in C$ and $B \in C$ and nothing else

Axiom of Unions

Let $\mathcal{C}$ be a set of sets. Then there is a set which contains all elements which belong to at least one set from $\mathcal{C}$, and nothing else

from lecture 3

Axiom of Power sets

For every set E, there is a set $\mathcal{P}$ consisting of precisely the subsets of E

that is $A \subset E$ if and only if $A \in \mathcal{P}$

from lecture 5

Axiom of infinity

There exists a set containing (as an element) $\emptyset$ and containing (as an element) the successor of each of its elements

Where does this get us?

Today, the standard axiomatization of set theory used throughout mathematics is

Zermelo - Fraenkel Set Theory

What is the history?

In 1908 Ernst Zermelo wrote a paper giving 7 axioms for set theory

His axioms were

Zermelo set theory

1. Axiom of extensionality (our axiom of extension)
2. Axiom of elementary sets (our axiom of the empty set and axiom of pairing together)
3. Axiom of separation (yet another name for the axiom of subset selection)
4. Axiom of the power set (same as ours)
5. Axiom of the union (same as ours)
6. Axiom of choice

next lecture

7. Axiom of infinity (his gave $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\}$ rather than our ω, but a very similar idea)

Zermelo's axiom of separation, like ours currently, was a bit vague on what sort of properties were allowed – the machinery of first order logic was not yet available

Also these axioms are not powerful enough to talk about bigger infinities

In 1922 Abraham Fraenkel and Thoralf Skolem
added two more axioms
The axiom of regularity or foundation
and
The axiom of replacement or substitution
and formalized the axiom of separation.
giving
Zermelo - Fraenkel set theory

Before we get into all that lets finish Zermelo set theory.

(2) Families

To state the axiom of choice we need to return to Halmos chapter 9 and define families

A family of sets is something like $\{A_i\}_{i\in I}$

There are collections of sets A_i each given an index i from some other set I

eg $A_n = \{n, 2n, 3n, 4n, \ldots\}$

Consider $\{A_n\}_{n\in\omega}$ This is a family of sets indexed by ω

To formalize this use ordered pairs let the family $\{A_n\}_{n\in\omega}$ be the set $\{(i, A_i) : | \, i\in\omega\}$ of ordered pairs.

That is a family is a function

$f: I \longrightarrow \{A_i : i\in\omega\}$ ← or any other set with all A_i as members

$f(i) = A_i$

eg Continuing the above eg $f: \omega \to \mathcal{P}(\omega)$

$f(n) = \{n, 2n, 3n, \ldots\}$

We have a formal notion of function in set theory, namely a function is the set of ordered pairs $(x, f(x))$

So define a family $\{A_i\}$ of subsets of X indexed by I to be a function $f : I \to \mathcal{P}(X)$

- we call I the index set
- we call an $i \in I$ an index
- we call the image of i, $f(i)$, (namely A_i) a term of the family

We can use families to define larger cartesian products.

Recall $X \times Y$ is $\{(x, y) \mid x \in X, y \in Y\}$

We can rewrite this using families

take any unordered pair $\boxed{\{a, b\} = I}$

make a family with I the index set

$f: I \to X \cup Y$ with the property $f(a) \in X$, $f(b) \in Y$

idea is f is essentially the same as $(f(a), f(b))$

let Z be the set of such families

Claim Z and $X \times Y$ are essentially the same

precisely, there is a one-to-one and onto map between them.

proof of claim let $g: Z \to X \times Y$

$g(f) = (f(a), f(b))$

check g is one-to-one Suppose $g(f_1) = g(f_2)$

want $f_1 = f_2$. $(f_1(a), f_1(b)) = (f_2(a), f_2(b))$

so $f_1(a) = f_2(a)$ and $f_1(b) = f_2(b)$

but f_1 and f_2 have domain $\{a, b\}$

so $f_1 = f_2$

$g: Z \to \underbrace{X \times Y}$

check g is onto. Take any $(x, y) \in X \times Y$.

let $f \in Z$ be given by $f(a) = x$ $f(b) = y$

then $g(f) = (f(a), f(b)) = (x, y)$ so (x, y) is in the image of g so g is onto $X \times Y$

How do we know Z is even a set?

need to write $Z = \{ f \in (X \cup Y)^{\{a,b\}} \mid f(a) \in X, f(b) \in Y \}$

Now we have the tools to define general cartesian products

Definition Let $\{Y_i\}$ be a family of sets indexed by I. Then the **cartesian product** of the family

$$\bigtimes_{i \in I} Y_i$$

is the set of all families

$$y : I \to \bigcup_{i \in I} Y_i$$

with $y(i) \in Y_i$

eg Let $I = \{a, b, c\}$

Let $Y_a = \{1, 2, 3\}$

Let $Y_b = \{4, 5\}$

Let $Y_c = \{6\}$

Then $\bigtimes_{i \in I} Y_i$ is the set of all families $y : \{a, b, c\} \to \{1, 2, 3, 4, 5, 6\}$

with $y(a) \in \{1,2,3\}$ $y(b) \in \{4,5\}$ $y(c) \in \{6\}$

What are the possibilities for y?

$y(a)$	$y(b)$	$y(c)$
(1,	4,	6)
(2,	4,	6)
(3,	4,	6)
(1,	5,	6)
(2,	5,	6)
(3,	5,	6)

3 Next time

The axiom of choice

Please read Halmos section 15

If you can easily bring a laptop (with wireless) please do