More constructions

Contents

1 Recall

Sequence If $\mathcal{B}$ is a class then the sequence class $\textsc{Seq}(\mathcal{B})$ is defined to be the infinite sum

$$\mathcal{A} = \textsc{Seq}(\mathcal{B}) = \mathcal{E} + \mathcal{B} + (\mathcal{B}\times\mathcal{B}) + (\mathcal{B}\times\mathcal{B}\times\mathcal{B}) + \dots$$

equivalently, $\mathcal{A} = \{(\beta_1, \beta_2, \dots, \beta_l) \text{ s.t. } \beta_j \in \mathcal{B}, l \geq 0\}$ This only works if $\mathcal{B}$ does not contain an element of size zero (a neutral element). Also $\alpha = (\beta_1, \beta_2, \dots, \beta_l) \Rightarrow |\alpha| = |\beta_1| + \dots + |\beta_l|$

$$\mathcal{A} = \textsc{Seq}(\mathcal{B}) \implies A(z) = \frac{1}{1-B(z)}$$

Notice that if we want a sequence that contains exactly k-objects or at least k objects then we might write $\textsc{Seq}_k(\mathcal{B}) = \mathcal{B}^k$ and $\textsc{Seq}_{\geq k}(\mathcal{B}) = \mathcal{B}^k \times \textsc{Seq}(\mathcal{B})$

Examples of specifications using sequence

Binary Words $\mathcal{W} = \textsc{Seq}(\mathcal{Z}_1 + \mathcal{Z}_0)$. Then, the ogf is $W(z) = \frac{1}{1-2z}$.

Positive integers Let $\mathcal{Z} = \{\circ\}$. Then $\mathcal{I} = \textsc{Seq}(\mathcal{Z}) = \{\epsilon, \circ, \circ\circ, \circ\circ\circ, \dots\}$. The OGF is $\frac{1}{1-z}$.

Interval covers Let $\mathcal{A} = \{\circ, \circ\!\!-\!\!\circ\}$. Then $\mathcal{B} = \textsc{Seq}(A)$ are the coverings of $[0, n]$ by intervals of length $1, 2$.

$$\epsilon, \circ, \circ\circ, \circ\!\!-\!\!\circ, \circ\circ\circ, \circ\,\circ\!\!-\!\!\circ, \circ\!\!-\!\!\circ\circ, \dots$$

The ogf is

$$B(z) = 1 + z + 2z^2 + 3z^3 + 5z^4 + \dots = \frac{1}{1-(z+z^2)}$$

Plane trees $\mathcal{T} = \mathcal{Z} \times \textsc{Seq}(\mathcal{T})$. Then the ogf satisfies

$$T(z) = \frac{z}{1-T(z)}.$$

Definition. A specification for an r-tuple of classes $\mathcal{A} = (\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(r)})$ is a set of r equations

$$\begin{aligned}\mathcal{A}^{(1)} &= \Phi_1(\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(r)})\\ &\vdots\\ \mathcal{A}^{(r)} &= \Phi_r(\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(r)})\end{aligned}$$

where each Φ_i is built using the admisible constructions we know as well as the neutral class $\mathcal{E}$ and atomic class $\mathcal{Z}$.

2 Advanced constructions

Next we define three more advanced constructions based upon equivalence classes of sequences: Cycles, Power sets, and multisets.

2.1 Cycles

The next constructions are a little trickier. We ultimately view them as equivalence classes of sequences. Think of a string of beads of 2 colours on a circular necklace. How many necklaces can be made using, say, 5 beads? Clearly 2 necklaces are the same if one is a cyclic permutation of the other. More generally, one can think of cycles as equivalence classes of sequences modulo circular shifts. We say that a cycle with k cycles is a k-cycle. For example, if we permit ourselves two bead colours, a and b, we consider the cycles on $\{a, b\}$:

$$1\text{ cycle} = \begin{cases} a \\ b \end{cases} \qquad 2\text{-cycle} = \begin{cases} aa \\ ab = ba \\ \qquad bb \end{cases}$$

$$3\text{-cycle} = \begin{cases} aaa \\ aab = aba = baa \\ abb = bba = bab \\ bbb \end{cases} \qquad 4\text{-cycle} = \begin{cases} aaaa \\ aaab = aaba = abaa = baaa \\ aabb = abba = bbaa = baab \\ abab = baba \\ abbb = bbba = bbab = babb \\ bbbb \end{cases}$$

Definition (Cycle construction)**.** Sequences modulo circular shifts define cycles which we denote $\textsc{Cyc}(\mathcal{B})$. More precisely

$$\textsc{Cyc}(\mathcal{B}) = (\textsc{Seq}(\mathcal{B}) - \mathcal{E})/\mathbf{S}$$

Where $(\beta_1, \ldots, \beta_l)S(\beta_1', \ldots, \beta_l')$ iff there is some circular shift so that β_j' is the same as β_{j+d} (being careful with $\mod l$). That is, $\beta_j' = \beta_{1+(j-1+d) \mod l}$.

These are defined using sequence so we insist that $B_0 = 0$.

Example. The class of all necklaces $\mathcal{N}$ with 5 possible colours of beads:

$$\mathcal{N} = \textsc{Cyc}(\mathcal{Z}_{red} + \mathcal{Z}_{blue} + \mathcal{Z}_{yellow} + \mathcal{Z}_{green} + \mathcal{Z}_{purple})$$

We will deal with generating functions in a moment.

2.2 Multisets

Next are multisets. Multisets are just like normal sets — order of elements does not matter, but now one can have repetitions of elements. eg $\{1, 1, 2, 3, 3, 3, 3, 7\}$. We only consider *finite* sets.

Definition. We define multisets as sequences of objects modulo permutations of the elements.

$$\textsc{MSet}(\mathcal{B}) = \textsc{Seq}(\mathcal{B})/\mathbf{R}$$

where $(\alpha_1, \ldots, \alpha_l)\mathbf{R}(\beta_1, \ldots, \beta_l)$ iff there is some permutation σ such that $\alpha_j = \beta_{\sigma(j)}$.

These are defined using sequence so we insist that $B_0 = 0$.

Exercise. Show that there are 5 multi-sets of size 4 in $\textsc{MSet}(a, b)$.

2.3 Powersets

Last we have powersets.

Definition. A powerset $\textsc{Pset}(\mathcal{B})$ of $\mathcal{B}$ is the set of all subsets of elements. Equivalently it is a multiset in which no repetitions are allowed; so $\textsc{Pset}(\mathcal{B}) \subset \textsc{Mset}(\mathcal{B})$.

These are defined using sequence so we insist that $B_0 = 0$.

Exercise. How many power sets are there of size 4 in $\textsc{Pset}(a, b)$?

2.4 Admissibility theorem for ogf

Each of these constructions are admissible, and has a direct generating function translation. It is beyond the scope of this course to derive them (although the power- and multi- set constructions are not that difficult). We list all of them in the following table.

Theorem (1.1)**.** *The constructions of union, cartesian product, sequence, powerset, multiset and cycle are all admissible. The operators are*

sum	$\mathcal{A} = \mathcal{B} + \mathcal{C}$	$A(z) = B(z) + C(z)$
cartesian product	$\mathcal{A} = \mathcal{B} \times \mathcal{C}$	$A(z) = B(z)C(z)$
sequence	$\mathcal{A} = \textsc{Seq}(\mathcal{B})$	$A(z) = \dfrac{1}{1 - B(z)}$
powerset	$\mathcal{A} = \textsc{Pset}(\mathcal{B})$	$A(z) = \prod_{n\geq 1}(1+z^n)^{B_n} = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} B(z^k)\right)$
multiset	$\mathcal{A} = \textsc{Mset}(\mathcal{B})$	$A(z) = \prod_{n\geq 1}(1-z^n)^{-B_n} = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} B(z^k)\right)$
cycle	$\mathcal{A} = \textsc{Cyc}(\mathcal{B})$	$A(z) = \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log \frac{1}{1 - B(z^k)}$

where for all but sum & cartesian product, it is assumed that $\mathcal{B}_0 = \varnothing$, and

$$\begin{aligned}\varphi(k) &= \textit{Euler totient function}\\ &= \textit{number of integers in } [1, k] \textit{ relatively prime to } k\end{aligned}$$

3 Examples of combinatorial specifications using the additional constructions

Lets see some examples of constructing combinatorial classes using these additional constructions.

3.1 Partitions $\mathcal{P}$

A partition of n is very similar to a composition (see lecture 6), in that it is a set of numbers that sum to n, but the ordering of the summands does not distinguish between partitions. For example, the two compositions 1+1+2 and 2+1+1 represent the same partition. As such, we take the convention of listing the summands in increasing order. The partitions of 4 are thus

$$\mathcal{P}_4 = \{1+1+1+1, 1+1+2, 2+2, 1+3, 4\}.$$

Remark, above we denoted compositions by a sequence, and the condition on partitions means that that paritions are compositions modulo a permutation of order of the elements. We have an operator for that: Multi-set.

$$\begin{aligned} 1+1+1+1 &\leftrightarrow \{\circ,\circ,\circ,\circ\} \\ 1+1+2 &\leftrightarrow \{\circ,\circ,\circ\!-\!\circ\} \quad (\leftarrow \text{This is equivalent to } \{\circ,\circ\!-\!\circ,\circ\}) \\ 2+2 &\leftrightarrow \{\circ\!-\!\circ,\circ\!-\!\circ\} \\ 1+3 &\leftrightarrow \{\circ,\circ\!-\!\circ\!-\!\circ\} \\ 4 &\leftrightarrow \{\circ\!-\!\circ\!-\!\circ\!-\!\circ\} \end{aligned}$$

We view partitions are as multisets of natural numbers:

$$\begin{aligned} \mathcal{P} &= \text{MSET}(I) \\ P(z) &= \prod_{n\geq 1}(1-z^n)^{-I_n} = \prod_{n\geq 1}(1-z^n)^{-1} \\ &= 1+z+2z^2+3z^3+5z^4+7z^5+11z^6+15z^7+22z^8+30z^9+\dots. \end{aligned}$$

There is no simple form for these coefficients, though it is not hard to compute them in polynomial time.

Exercise. Why is the subset of partitions comprised of partitions with unique parts given by $\text{PSET}(\mathcal{I})$? (There are only two of size 4) What is the generating function?

We can ask some of the same questions as we did with compositions.

Partitions		
Type	Spec	ogf
all	$\text{MSET}(\text{SEQ}_{\geq 1}(\mathcal{Z}))$	$\prod_{n\geq 1}(1-z^n)^{-1}$
parts $\leq r$	$\text{MSET}(\text{SEQ}_{1\dots r}(\mathcal{Z}))$	$\prod_{n=1}^{r}(1-z^n)^{-1}$
$\leq k$ parts	$\text{MSET}_{\leq k}(\text{SEQ}_{\geq 1}(\mathcal{Z})) \cong \text{MSET}(\text{SEQ}_{1\dots k}(\mathcal{Z}))$	$\prod_{n=1}^{k}(1-z^n)^{-1}$
distinct parts	$\text{PSET}(\text{SEQ}_{\geq 1}(\mathcal{Z}))$	$\prod_{n\geq 1}(1+z^n)$

3.2 Non-plane trees

Next we consider tree structures in which the order of the children doesn't matter. See lecture 6 for plane trees. One can do a similar decomposition — delete the root node and see what is left over. Instead of getting an ordered sequence of offspring, one will get a set of offspring.

Rooted plane trees		
Type	Spec	ogf
general	$\mathcal{G} = \mathcal{Z}\text{SEQ}(G)$	$\frac{1-\sqrt{1-4z}}{2}$
binary	$\mathcal{B} = \epsilon + \mathcal{Z}\times\mathcal{B}\times\mathcal{B}$	$\frac{1-\sqrt{1-4z}}{2z}$
simple	$\mathcal{T} = \mathcal{Z}\text{SEQ}_{\Omega}(\mathcal{T})$	$T(z) = z\phi(T(z))$
Rooted non-plane		
Type	Spec	ogf
general	$\mathcal{H} = \mathcal{Z}\times\text{MSET}(\mathcal{H})$	$H(z) = z\mathbf{Exp}(H(z))$
binary	$\mathcal{V} = \mathcal{Z}\times\text{MSET}_2(\mathcal{V})$	$U(z) = z + (V(z)^2 + V(z^2))/2$
simple	$\mathcal{U} = \mathcal{Z}\times\text{MSET}_{\Omega}(\mathcal{U})$	messy

In the case of non-plane binary trees, when you delete the root one gets a pair of trees (τ_1, τ_2). These pairs are nearly counted by $U(z)^2$, but since $(\tau_1, \tau_2) \equiv (\tau_2, \tau_1)$ we should have $U(z)^2/2$. This is still not quite right, because it gives the wrong answer when $\tau_1 = \tau_2$ (count is half what it should be). Thus we must add back in half the ogf of such pairs $U(z^2)/2$. Thus

$$U(z) = z(1 + V(z)/2 + V(z^2)/2)$$

Clearly we don't want to have to do this for the general case, but thankfully we have already done all the hard work

$$\mathcal{H} = \mathcal{Z} \times \textsc{Mset}(\mathcal{H})$$

Delete the root vertex of a non-plane rooted tree and you get a multi-set of trees. Thus

$$\begin{aligned} H(z) &= z \exp\left(\sum_{k\geq 1} \frac{1}{k} H(z^k)\right) \\ &= \prod_{n\geq 1} (1 - z^n)^{-H_n} \\ &= z + z^2 + 2z^3 + 4z^4 + 9z^5 + 20z^6 + 48z^7 + \dots \end{aligned}$$

This was first done by Cayley in 1850-something and there is no nice closed form for the gf or the coefficients.

Again one can consider similar non-plane trees with restricted vertex out-degrees and things are not too much uglier. If outdegrees are constrained to lie in a set Ω then we have

Lemma. *Let Ω be a finite subset of the non-negative integers that contains zero. Then the ogf, $U(z)$, of non-plane rooted trees whose vertices have outdegrees constrained to lie in Ω is given by*

$$\mathcal{U} = \mathcal{Z} \times \textsc{Mset}_{\Omega}(\mathcal{U})$$
$$U(z) = \Phi(U(z), U(z^2), \dots,)$$

where $\Phi(\vec{u})$ is a polynomial given by

$$\Phi(U(z), U(z^2), \dots,) = \sum_{\omega\in\Omega} [u^{\omega}] \exp\left(\sum_{k\geq 1} \frac{1}{k} U(z^k)\right)$$

So - this is not so pretty, but it is still do-able. Eg

$$\Phi = 1 + (U(z) + U(z^2))/2 \qquad \text{binary}$$
$$\Phi = 1 + (U(z)^3/6 + U(z)U(z^2)/2 + U(z^3)/3) \qquad \text{ternary}$$