Friday September 20 Lecture Notes

1 Functors

Definition Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* (or covariant) F is a function that assigns each $C \in \mathrm{Obj}(\mathcal{C})$ an object $F(C) \in \mathrm{Obj}(\mathcal{D})$ and to each $f : A \to B$ in $\mathcal{C}$, a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$, satisfying:

For all $A \in \mathrm{Obj}(\mathcal{C})$, $F(1_A) = 1_{FA}$.
Whenever fg is defined, $F(fg) = F(f)F(g)$.

e.g. If $\mathcal{C}$ is a category, then there exists an identity functor $1_{\mathcal{C}}$ s.t. $1_{\mathcal{C}}(C) = C$ for $C \in \mathrm{Obj}(\mathcal{C})$ and for every morphism f of $\mathcal{C}$, $1_{\mathcal{C}}(f) = f$.

For any category from universal algebra we have "forgetful" functors.

e.g. Take $F : \mathrm{Grp} \to \mathrm{Cat}$ of *monoids* $(\cdot, 1)$. Then $F(G)$ is a group viewed as a monoid and $F(f)$ is a group homomorphism f viewed as a monoid homomorphism.

e.g. If $\mathcal{C}$ is any universal algebra category, then
$F : \mathcal{C} \to \mathrm{Sets}$
$F(C)$ is the underlying sets of $\mathcal{C}$
$F(f)$ is a morphism

e.g. Let $\mathcal{C}$ be a category. Take $A \in \mathrm{Obj}(C)$. Then if we define a covariant Hom functor, $\mathrm{Hom}(A, \_) : \mathcal{C} \to \mathrm{Sets}$, defined by $\mathrm{Hom}(A, \_)(B) = \mathrm{Hom}(A, B)$ for all $B \in \mathrm{Obj}(\mathcal{C})$ and $f : B \to C$, then $\mathrm{Hom}(A, \_)(f) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ with $g \mapsto fg$ (we denote $\mathrm{Hom}(A, \_)$ by f_*). Let us check if f_* is a functor:

Take $B \in \mathrm{Obj}(\mathcal{C})$. Then $\mathrm{Hom}(A, \_)(1_B) = (1_B)_* : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B)$ and for $g \in \mathrm{Hom}(A, B)$, $(1_B)_*(g) = 1_B g = g$. So $(1_B)_* = 1_{\mathrm{Hom}(A,B)}$.

Take $B \xrightarrow{f} C \xrightarrow{g} D$. Certainly, $\mathrm{Hom}(A, B) \xrightarrow{f_*} \mathrm{Hom}(A, C) \xrightarrow{g_*} \mathrm{Hom}(A, D)$. Now take $h \in \mathrm{Hom}(A, B)$. Then $f_*(g_*(h)) = fgh = (fg)h = (fg)_* h = \mathrm{Hom}(A, \_)(fg)$.

A few observations:

Proposition Functors preserve isomorphisms at the level of morphisms, i.e., if $T : \mathcal{C} \to \mathcal{D}$ and $f : A \to B$ is an isomorphism in $\mathcal{C}$, then $T(f)$ is an isomorphism in $\mathcal{D}$.
Proof Functors preserve compositions and identities. Say $g : B \to A$ with $fg = 1_B$, and $fg = 1_B$. Then $T(fg) = T(f)T(g)$, but $T(fg) = T(1_B) = 1_{T(B)}$. A similar argument works for $gf = 1_A$, and we are done.

Definition Two categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic (as categories) if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with $F(G) = 1_{\mathcal{D}}$ and $G(F) = 1_{\mathcal{C}}$ (where the composition of functors is just a composition on objects and a composition on maps).

e.g. Given a ring R, let R^{op} denote R but with multiplication defined backwards: $r_1 \cdot_{R^{\mathrm{op}}} r_2 = r_2 \cdot_R r_1$ for all $r_1, r_2 \in R$ or $r_1, r_2 \in R^{op}$ (because they have the same underlying set). Then R-mod is isomorphic as a category to mod-R^{op}, the category of right modules over R^{op}.

2 Covariant Functors

Definition If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a *covariant* functor $T : \mathcal{C} \to \mathcal{D}$ is a functor taking $C \in \mathrm{Obj}(\mathcal{C})$ to $T(C) \in \mathrm{Obj}(\mathcal{D})$, and $f : C \to D$ in $\mathcal{C}$ to $T(f) : T(D) \to T(C)$ in $\mathcal{D}$, satisfying:

$T(1_A) = 1_{T(A)}$ for all $A \in \mathrm{Obj}(\mathcal{C})$

If $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$, then $T(C) \xrightarrow{T(g)} T(B) \xrightarrow{T(g)} T(A)$ in $\mathcal{D}$, i.e., $T(gf) = T(f)T(g)$.

e.g. Let $\mathcal{C}$ be a category. Then the covariant Hom functor $\mathrm{Hom}(\_, B) : \mathcal{C} \to \mathrm{Sets}$, with $B \in \mathrm{Obj}(\mathcal{C})$, is defined by $\mathrm{Hom}(\_, B)(f) : \mathrm{Hom}(D, B) \to \mathrm{Hom}(C, B)$ with $g \mapsto gf$. We write $\mathrm{Hom}(\_, B)(f) = f^*$.

3 Natural Transformations

Definition Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Then a *natural* transformation (or a morphism of functors) τ from F to G, $\tau : F \to G$, is a functor that assigns each $C \in \mathrm{Obj}(\mathcal{C})$ a morphism of $\mathcal{D}$ with $\tau_C : F(C) \to G(C)$ s.t. for all $F : C \to C'$, the following diagram commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{\tau_C} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow[\tau_{C'}]{} & G(C') \end{array}$$

e.g. Let F : Grp $\to$ Sets be the "forgetful" functor and let S : Grp $\to$ Sets be the "squaring" functor defined by $S(G) = G \times G$ (viewed as a set) and $S(f : G \to H) = f \times f : G \times G \to H \times H$. So the group multiplication on G is a functor $\tau_G : G \times G \to G$.

Claim τ is a natural transformation.
Take group homomorphism $f : G \to G'$ such that

$$\begin{array}{ccccccccc}
G & \times & G & = & S(G) & \xrightarrow{\tau_G} & F(G) & = & G \\
\downarrow f & & \downarrow f & & \downarrow S(f) & & \downarrow F(f) & & \downarrow f \\
G & \times & G' & = & S(G') & \xrightarrow[\tau_{G'}]{} & F(C') & = & G'
\end{array}$$

This diagram says that $f(x)f(y) = f(xy)$, i.e., f is a group homomorphism.

Definition A natural transformation $\tau : F \to F'$ is a *natural* isomorphism if each τ_A is an isomorphism. In this case we say F and F' are *naturally* isomorphic and write $F \simeq F'$. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exist $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ s.t. $F(G) \simeq 1_{\mathcal{D}}$ and $G(F) \simeq 1_{\mathcal{C}}$.

4 Finitely Generated Modules

Definition Let R be a ring and let M be a left module over R. Let $R_a = \{ra : r \in R\}$ denote a cyclic module generated by $a \in M$. Then M is *cyclic* if $M = R_a$ for some $a \in M$.

e.g. Viewing R as a module over itself, $R = R \cdot 1$ is cyclic, and the cyclic submodules of R are exactly the principle ideals.

Proposition A module M is cyclic if and only if $M \cong R/L$ where L is some left ideal of R.
Proof Suppose M is cyclic, i.e., $M = R_a$. Take $f_a : R \to M$ with $r \mapsto ra$ where $\ker f_a = \text{Ann}_R(a) = L$. Note that f_a is onto because $M = R_a$ and so by the First Isomorphism Theorem, $M \cong R/L$. Now assume $M \cong R/L$. Take any coset $r + L = r(1 + L)$ (conversely, any $r(1 + L) = r + L$ is a coset of L). So $R/L = R(1 + L) = R(a)$, and we are done.

If R is a PID, i.e., every ideal of R is principal, then every cyclic module has the form R/R_d for some $d \in R$.

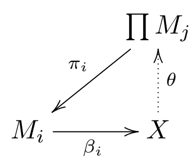
Definition Let $S = \{a_i\}_{i \in I}$ be a subset of a module M. We say S *spans* M if every element of M can be written as as a finite sum $\sum_{i \in I} r_i a_i$. Moreover, M is *finitely generated* if it is spanned by a finite set (in this case $M = \sum_{i=1}^{t} r_i a_i$).

5 Direct Sums and Cartesian Products of Modules

We know they should be: the categorical coproduct and product, respectively, but we need to show they exist. Cartesian product works for any universal algebraic category.

Definition Let $\{M_i\}_{i \in I}$ be R-modules and let $\prod_{i \in I} M_i$ be the set (Cartesian product) with $+$ component-wise and $r((a_i)_{i \in I}) = (ra_i)_{i \in I}$.

This is a module since all the identities/axioms hold component-wise and so hold in $\prod_{i \in I} M_i$. If we take this with projections $\pi_i : \prod M_j \to M_i$ with $(a_j)_{j \in I} \mapsto a_i$. We need to check this satisfies universal properties of products:



where we define θ by $\theta(x) = (\beta_i(x))_{i \in I}$, and this is the unique map which works.

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