Math 800 Commutative Algebra Notes: October 9

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For these sections will always assume that C is a commutative ring.

1 Algebras

Definition 1.1. A *C-algebra* is a ring, R, which is also a C-module with the property:

$$c(r_1r_2) = (cr_1)r_2 = r_1(cr_2)$$

for all $c \in C$ and $r_1, r_2 \in R$.

A C-algebra is a universal algebra object since rings and C-modules are and only one more axiom is added. This means that we get the definitions of subalgebra; algebra homomorphism, isomorphism, etc. automatically.

Note that when checking if a map is an algebra homomorphism it is only necessary that the map be a ring homomorphism and a module homomorphism.

We also automatically get the three isomorphism theorems. The correct congruence for the isomorphism theorems is moding out by ideals of the algebra.
Example 1.2. Any ring is a $\mathbb{Z}$-module with

$$nr = \begin{cases} \sum_{i=1}^{n} r & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ \sum_{i=1}^{-n} (\text{-}r) & \text{if } n < 0. \end{cases}$$

This also gives R as a $\mathbb{Z}$-algebra.
Example 1.3. If $\lambda_1, \ldots, \lambda_n$ are commuting indeterminates then $C[\lambda_1, \ldots, \lambda_n]$ is a C-algebra .
Definition 1.4. Let R be a C-algebra and $S \subseteq R$ then

$$C[S] := \left\{ \sum_{\text{finite}} c_{i_1 \cdots i_n} s_1^{i_1} \cdots s_n^{i_n} : c_{i_1 \cdots i_n} \in C, s_j \in S \right\}$$

or equivalently $C[S]$ is the intersection of all subalgebras of R that contain S.
Definition 1.5. If R is a ring then

$$\text{Cent}(R) := \{r \in R : rs = sr \ \forall s \in R\}$$

is called the center of R.

Example 1.6. If R is a ring and $C \subseteq \text{Cent}(R)$ then R is a C-algebra where module multiplication is ring multiplication in R.
Proposition 1.7. *Let R be a C-algebra then*

- *there exists an algebra homomorphism*

$$\begin{array}{rccc} \phi: & C & \to & \text{Cent}(R) \\ & c & \mapsto & c1_R \end{array}$$

  and

- *let $C' = \phi(C)$ then R is a C'-algebra with $cr = \phi(c)r$.*

Proof. See Proposition 5.4 of Rowen. □

Definition 1.8. Let ϕ be as in the previous proposition. We say R is *faithful* if $\ker \phi = 0$.

By the first isomorphism theorem, for a faithful algebra $C' \cong {}^{C}/_{\ker \phi} \cong C$.
Lemma 1.9 (Substitution lemma)**.** *Suppose $f : R \to T$ is a C-algebra homomorphism then for all $a \in \text{Cent}(T)$ there exists a C-algebra homomorphism, $\tilde{f} : R[\lambda] \to T$ with $\tilde{f}(\lambda) = a$ and $\tilde{f}|_R = f$, where λ is an indeterminate.*

Proof. Define $\tilde{f}(\sum r_i \lambda^i) = \sum f(r_i) a^i$.

See Lemma 5.6 of Rowen. □

By induction we get the following proposition.
Proposition 1.10 (Substitution proposition)**.** *Let R be a C-algebra and $a_1, \ldots, a_n \in \text{Cent}(R)$ then there exists $f : C[\lambda_1, \ldots, \lambda_n] \to R$ given by $f : \lambda_i \to a_i$, where $\lambda_1, \ldots, \lambda_n$ are commuting indeterminants.*

2 Affine algebras

Definition 2.1. An *affine algebra* is a commutative algebra, R, over a field, F, such that $R = {}^{F[a_1,\ldots,a_n]}/_A$ for some $a_i \in R$, $n \in \mathbb{N}$.

We say R is an *affine domain* if it is an affine algebra and an integral domain.
Proposition 2.2. *R is an affine algebra if and only if $R \cong {}^{F[\lambda_1,\ldots,\lambda_n]}/_A$, for some $\lambda_1, \ldots, \lambda_n$, commuting indeterminants and A, ideal of $F[\lambda_1, \ldots, \lambda_n]$.*

The next example shows that subalgebras of affine algebras are not necessarily affine.
Example 2.3. We know that $F[\lambda_1, \lambda_2]$ is an affine algebra by the previous proposition. Let $S = F + \lambda_1 F[\lambda_1, \lambda_2]$. Then S is an algebra that cannot be generated by finitely many elements.

We shall work up to the following main theorem.
Theorem 2.4 (Theorem A)**.** *An affine domain, R, is a field if and only if R is algebraic over F.*

Recall that R is an F-algebra. We say that $a \in R$ is *algebraic* over F if a is a root of a polynomial in $F[\lambda]$. R is *algebraic* over F if every element of R is algebraic over F.

If $F[a]$ is an F vector space then $\{1, a, a^2, \ldots\}$ spans $F[a]$. If $F[a]$ is finite dimensional then there is a linear dependence in $\{1, a, a^2, \ldots\}$ and so a satisfies a polynomial in $F[\lambda]$. Also if a is algebraic then it solves some polynomial of some degree n and so $\{1, a, \ldots, a^{n-1}\}$ spans $F[a]$. This implies that to check that $R = F[a_1, \ldots, a_n]$ is algebraic we only need to check that its generators are algebraic.

Definition 2.5. Let C be an integral domain then the *field of fractions* of C, $\mathrm{Q}(C)$ is the set of equivalence classes of pairs (a, b), $a, b \in C$ and $b \neq 0$ (written $\frac{a}{b}$) under the equivalence

$$(a, b) \sim (c, d) \iff ad = bc.$$

$\mathrm{Q}(C)$ is a field using $+, \cdot$ defined for fractions and C can be embedded in $\mathrm{Q}(C)$ via $c \mapsto (c, 1)$.

The following lemmas will be the machinery of the induction of Theorem A.

Lemma 2.6. *If R is an F-algebraic domain and $a \in R$ then if $K = \mathrm{Q}(F[a])$ is affine over F then a is algebraic over F and $K = F[a]$.*

Proof. See Remark 5.13 of Rowen □

Lemma 2.7. *Let K be a commutative ring, R be a K-algebra that is R free with base B as a K-module and be M be a subring of K. If B spans R over H then $H = K$.*

Proof. See Remark 5.14 of Rowen □

Proposition 2.8 (Artin-Tate Lemma)**.** *If $R = F[a_1, \ldots, a_n]$ is an affine F-algebra and K is a subring of R such that R is finitely generated as a K-module then K is affine over F.*

Proof. Next time. □