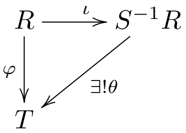
We are talking about localization. We will be mostly concerned with the case of a commutative ring, but we will try to use as much generality as possible.

We first define what a monoid is. A monoid is a set A with a operation $A \times A \to A$ that is associative, and has a identity element. Now given a ring $R \neq 0$ and and a sub monoid S of R contained in $\text{Cent}(R)$ (This is a fancy way of saying that S is closed under multiplication and that $1 \in S$. In the literature for commutative rings this is often referred to as a multiplicative set) that does not contain 0 we can define the localization of R at S is a a pair $(\iota, S^{-1}R)$ where $\iota :\to S^{-1}R$ is a algebra homomorphism and the following universal property. If $\varphi : R \to T$ is an algebra morphism such that $\varphi(s)$ is invertible for all $s \in S$ then there is a unique morphism $\theta : S^{-1}R \to T$ such that the following diagram commutes.



Now, as usual if we define an object via universal property, we have to show that such a object exists.

Let $S^{-1}R$ be the set $\{(r, s\} : r \in R, s \in S\}$ equipped by the relation $(r_1, s_1) \sim (r_2, s_2) \iff \exists s_3 \in S$ s.t. $s_3 r_1 s_2 = s_3 r_2 s_1$ or that $s_3(r_1 s_2 - r_2 s_1) = 0$.

It is a fact that the relation $\sim$ defined above is a equivalence relation. This is proven in rowen page 226 Theorem 8.2. Since it is in the book I will omit the proof. From now on we will use the notation that $(r, s) = \frac{r}{s}$.

Now define operations on $S^{-1}R$ as follows.

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + r_2 s_1}{s_1 s_2}$$

$$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$$

Again it is true that these operations are well defined. The proof of such things is standard and omitted.

Now define $\iota(r) = \frac{r}{1}$. Then, ι is a ring homomorphism. Indeed, let

$$\iota(r_1 + r_2) = \frac{r_1 + r_2}{1} = \frac{r_1}{1} + \frac{r_2}{1} = \iota(r_1) + \iota(r_2)$$

because of how we defined addition. Similarly we know that

$$\iota(r_1 r_2) = \frac{r_1 r_2}{1} = \iota(r_1)\iota(r_2)$$

because of how we defined multiplication. Finally, it is clear that $\frac{1}{1}$ is the identity of $S^{-1}R$ so $\iota(1)$ is the identity. If we restrict ι to the center of R we see that

$$\iota\mid_{\text{Cent}(R)}(R) \subseteq \text{Cent}(S^{-1}R)$$

since given $\dfrac{r_1}{s_1}$ and $r \in \text{Cent}(R)$ we have

$$\iota(r)\frac{r_1}{s_1} = \frac{rr_1}{s_1} = \frac{r_1 r}{s_1} = \frac{r_1}{s_1}\iota(r)$$

So this makes $S^{-1}R$ a $\text{Cent}(R)$ algebra and in fact a $\text{Cent}(R)$ algebra morphism. Finally, one can compute $\ker\iota$. We know that $\frac{r}{1} = \frac{0}{1} \iff \exists s \in S$ such that $sr = 0$. So

$$\ker\iota = \{r \in R : \exists s \in S \text{ s.t.} sr = 0\}$$

OK, so we have defined what localization, but what does it mean. In one sense it is a generalization of the fraction field. One can see this as follows. If R is a commutative domain then take $S = \{r \in R : r \neq 0\}$. Then $S^{-1}R$ is the field of fractions of R. A few notes.

We have set it up so that $0 \notin S$. This means that if $s_1, s_2 \in S$ then $s_1 s_2 \neq 0$ because S is multiplicatively closed. A natural question at this point is as follows. When is ι an injection? That is, when can we embed R into the localization, This is easy. Namely if $s \in S$ and s is a zero divisor. That is, there is some $r \in R$ such that $rs = sr = 0$. Then we know that $\iota(r) = 0$ by our earlier work. That is, ι is an embedding iff S contains no zero divisors. So in fact we obtain the following.

Cor: If R is a integral domain then $S^{-1}R$ is a integral domain and ι is an embedding.

Proof: For the first claim,

$$\frac{r_1}{s_1}\frac{r_2}{s_2} = 0 \iff \exists s \in S \text{ s.t.} sr_1 r_2 = 0$$

That is we have $(sr_1)r_2 = 0$. Since R is a integral domain we know that $r_2 = 0$ or $sr_1 = 0$. But if $r_2 \neq 0$ then $r_1 s = 0$ and since $s \neq 0$ this means $r_1 = 0$ so $\dfrac{r_1}{s_1} = \frac{0}{s_1} = 0$ or $\dfrac{r_2}{s_2} = \frac{0}{s_2} = 0$. So $S^{-1}R$ is a domain. Furthermore, S cannot contain any zero divisors so by our earlier remarks ι is a injection.■

From now on I will assume all rings are commutative.

Continuing on, we can see that the largest (with respect to inclusion) multiplicative set such ι is an injection is the set of all non-zero divisors. One might also wonder how different multiplcative sets behave under localization. For those interested, there are some results in Atiyah Mcdonalds introduction to commutative algebra. For example the following holds. (The proofs are not so bad to work out. But I will omit them)

Lemma: S, T are multiplicative subsets of R. Let U be the image of T in $S^{-1}R$. Then $U^{-1}(S^{-1}R) \cong (ST)^{-1}R$

Lemma: Let Σ be the set of all multiplicative subsets of R. Then Σ has maximal elements and S is a maximal element of $\Sigma \iff R - S$ is a minimal prime ideal.

One might also wonder if $S \subseteq T$ are multiplicative sets, when is $S^{-1}R \cong T^{-1}R$. This leads to the notion of a saturated multiplicative set. Again such questions are addressed in the exercises of Atiyah-Mcdonald as above.

One of the reasons we think that localization is interesting is because the ideal theory of $S^{-1}R$ is a simplified version of the ideal theory of R. Given a set $A \subseteq R$ and S a multiplicative set we define

$$S^{-1}A := \{\frac{a}{s} : s \in S, a \in A\}$$

Now let I be an ideal of R.

Lemma: $S^{-1}I$ is a ideal of $S^{-1}R$.

Proof: If $i_1, i_2 \in I$ and $s_1, s_2 \in S$ then $\frac{i_1}{s_1} + \frac{i_2}{s_2} = \frac{i_1 s_2 + i_2 s_1}{s_1 s_2} \in S^{-1}I$ as I is an ideal. Similarly, $\frac{r}{s}\frac{i_1}{s_1} = \frac{ri_1}{ss_1} \in S^{-1}I$ for all $\frac{r}{s} \in S^{-1}I$. ■

Lemma: $\frac{1}{1} \in S^{-1}I \iff S \cap I \neq \emptyset$.

Proof: $\frac{1}{1} \in S^{-1}I$ means that there is some $i \in I$ and $s_1 \in S$ with $\frac{i}{s_1} = \frac{1}{1}$. That is, there is some $s_2 \in S$ with $s_2 i = s_1$. Since $s_2 i \in I$ we get that $s_1 \in I$ as desired. Conversely if $s \in I \cap S$ then

$$\frac{1}{s} \cdot \frac{s}{1} = \frac{1}{1} \in S^{-1}I$$

■

We now come to one of the main results.

Theorem:

(i) If I is a ideal of R and $S \cap R = \emptyset$ then $S^{-1}I$ is a proper ideal of $S^{-1}R$

(ii)Let J be a proper ideal of $S^{-1}R$. Then $J = S^{-1}I$ for a suitable ideal I of R . In fact $I = \{r \in R : \frac{r}{1} \in I\}$

(iii) Let $\mathcal{L}(S^{-1}R)$ be the ideals of $S^{-1}R$ and $\mathcal{L}(R)$ the ideals of R. Define

$$\Phi : \mathcal{L}(S^{-1}R) \to \mathcal{L}(R)$$

by $\Phi(J) = \{r \in R : \frac{r}{1} \in J\}$. Then Φ is an injective mapping that preserves inclusions. Furthermore, the image of Φ is contained in the ideals of R that are disjoint from S.

(iv) Let $\Psi : \mathcal{L}(R) \to \mathcal{L}(S^{-1}R)$ be the map $\Psi(I) = S^{-1}I$. Then $\Psi\Phi$ is the identity.

Proof: For (i) we know from the above lemmas that $S^{-1}I$ is an ideal and that $S^{-1}I$ is proper if $S \cap I$ is empty.

(ii) Let J be an ideal of $S^{-1}R$. I will show that $\Phi(J)$ as defined in the theorem is an ideal. Let $r_1, r_2 \in \Phi(J)$. Then $\frac{r_1}{1}, \frac{r_2}{1} \in J$. That is, $\frac{r_1}{1} + \frac{r_2}{1} =$

$\frac{r_1+r_2}{1} \in J$. This means that $r_1 + r_2 \in \Phi(J)$. If $r \in R$ and r_1 is as before we have $\frac{r}{1} \cdot \frac{r_1}{1} = \frac{rr_1}{1} \in J$ so that $rr_1 \in \Phi(J)$ so $\Phi(J)$ is an ideal. It remains to show that $\Psi\Phi(J) = J$.We have that

$$\Psi\Phi(J) = \{\frac{r}{s} : s \in S, r \in \Phi(J)\} = \{\frac{r}{s} : s \in S, \frac{r}{1} \in J\}$$

Note that if $\frac{r}{s} \in J$ then $\frac{s}{1}\frac{r}{s} = \frac{r}{1} \in J$. So $\frac{r}{s} \in \Psi\Phi(J)$ which means $J \subseteq \Psi\Phi(J)$ Conversely, if $\frac{r}{s} \in \Psi\Phi(J)$ then $\frac{r}{1} \in J$ so $\frac{1}{s} \cdot \frac{r}{1} = \frac{r}{s} \in J$ and equality prevails.

So $\Psi\Phi(J) = J$ as desired.

(iii) We have shown that $\Phi : \mathcal{L}(S^{-1}R) \to \mathcal{L}(R)$. We know that Φ is injective because it has a left inverse, namely Ψ from the proof of part 2. It remain to show that Φ preserves inclusions.

Namely if $J_1 \subseteq J_2$ then $\Phi(J_1) = \{r : \frac{r}{1} \in J_1\}$ but if $\frac{r}{1} \in J_1$ we know that $\frac{r}{1} \in J_2$ so that $r \in \Phi(J_2)$ so that $\Phi(J_1) \subseteq \Phi(J_2)$. Furthermore, since Φ is injective, it preserves strict inclusions.

(iv) We know that Ψ is a map from $\Psi : \mathcal{L}(R) \to \mathcal{L}(S^{-1}R)$ and that $\Psi\Phi$ is the identity by our earlier work.

■

Corr:

If R is Noetherian (respectively Artinian) then so is $S^{-1}R$)

This is immediate as we can translate any chain of ideals in $S^{-1}R$ to a chain of ideals in R that respects inclusions.

Corr: $K \dim S^{-1}R \leq K \dim R$

For the same reason as above.

I will now show that it is not the case that $\Phi\Psi$ is the identity.

Take $R = \mathbb{Z}$ and $S = \{2^n : n \geq 0\}$

Then $S^{-1}3\mathbb{Z} = \{\frac{3m}{2^n} : m \in \mathbb{Z}, n \geq 0\}$. On the other hand, $S^{-1}6\mathbb{Z} = \{\frac{6m}{2^n} : m \in \mathbb{Z}, n \geq 0\}$.

It is straightforward to show the sets are equal. Namely given $\dfrac{3m}{2^n}$. We have $\dfrac{3m}{2^n} = \dfrac{2}{2}\dfrac{3m}{2^n} = \dfrac{6m}{2^{n+1}} \in S^{-1}6\mathbb{Z}$ in $S^{-1}\mathbb{Z}$. On the other hand, given $\dfrac{6m}{2^n}$ we have $\frac{6m}{2^m} = \frac{3(2m)}{2^n} \in S^{-1}3\mathbb{Z}$. This gives us the desired equality.It follows that Φ is not a surjection. Furthermore, this example is another case of one of the most useful uses of localization. Namely, given an element $f \in A$ that is not nil potent. That is, $f^n \neq 0$ for any n (A is the ring in question) Then set $S = \{f^n : n \geq 0\}$. Then S is multiplicative and the ring $A_f = S^{-1}A$ is very useful. For example, $\text{Spec}(A_f) = \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \cap S\} = \emptyset$. That is, $\text{Spec}(A_f)$ are those prime ideals of A that do not contain f.

I will now give a tentative argument for why one might give localization its name. I will assume familiarity with localizing at a prime element covered in the next lecture. Consider a complex function of a single variable f. One might wonder if f is continuous or even better analytic. To check this, it suffices to check at every single point that the function is continuous or analytic. That is, local information (with respect to the topology of $\mathbb{C}$) is important. Now take $f(z)$ to be a polynomial over the complex numbers. If we choose a point z_0 such that $f(z_0) \neq 0$ then local to z_0 that is, in some small ball around z_0 we can invert f. That is, $\frac{1}{f(z)}$ is a perfectly good analytic function. In the algebraic case, let A be a ring and $a \in A$. Now define $f_a : \mathrm{Spec}(A) \to \cup_{\mathfrak{p} \in \mathrm{Spec}(A)} A/\mathfrak{p}$. By

$$f_a(\mathfrak{p}) \quad = \quad a \mod \mathfrak{p}$$

. That is, we regard elements of A as functions on the spectrum. Now, f_a is certainly a strange function, taking values in many different rings. But notice that if $a \notin \mathfrak{p}$ then $f_a(\mathfrak{p}) \neq 0$. So $a \in A - \mathfrak{p}$. So a is in fact invertible in $A_{\mathfrak{p}}$. If we identify a with f_a this gives, at least some evidence that $A_{\mathfrak{p}}$ could be regarded as lying close to $\mathfrak{p}$. Since functions not vanishing at $\mathfrak{p}$ are invertible in $A_{\mathfrak{p}}$.