Math 800 Commutative Algebra Notes: September 11

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1 Universal algebra (con't)

Last time we saw the three isomorphism theorems for universal algebra. Now we shall translate them into their ring theory counterparts:

The first isomorphism theorem is straightforward to translate:
Theorem 1.1 (1st isomorphism theorem for rings)**.** *Let $f : A \to B$ be a homomorphism of rings. Then there exists an injective map $g : A/Ker(f) \to B$ such that:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {}_{\nu}\searrow \quad \nearrow_{g} & \\ & A/Ker(f) & \end{array}$$

commutes and if f is onto then g is onto.

The second states:
Theorem 1.2 (2nd isomorphism theorem for rings)**.** *Let A be a ring and I and J be ideals of A with $J \subseteq I$ then*

$$(A/J)/(I/J) \cong A/I.$$

In particular there is a bijection between the set of ideals of A/I and the set of ideals of A containing I.

Now we shall show how this relates to the 2nd isomorphism theorem for universal algebra.

For a ring A and θ a congruence over A we consider $I = \{0\}/\theta$. Take $a, b \in I$ then $a\theta 0$ and $b\theta 0$ so $(a+b)\theta 0$ and $a + b \in I$. Also if $a \in I$ and $r \in R$ then $a\theta 0$ and $r\theta r$ so $ra\theta 0$ and $ra \in I$. Therefore I is an ideal.

Conversely if I is an ideal we can define θ by $a\theta b$ if and only if $a - b \in \theta$. It can then be shown that θ is a congruence.

To translate the 3rd we need to translate $\theta|_B$ and B^θ for a subring, B, of A and congruence, θ, of A.

Taking $I = \{0\}/\theta$, we can see that $I \cap B = \{0\}/\theta|_B$.

By definition B^θ is the set of all equivalence classes of A which contain an element of B. This means we want all the elements in each $b + I$ for $b \in B$. In other words $B^\theta = B + I$.

We also note that $\{0\}/\theta|_{B^\theta} = \{0\}/\theta$ because we only took whole equivalence classes.

So we get:
Theorem 1.3 (3rd isomorphism theorem for rings)**.** *Let A be a ring, I be an ideal of A and B be a subring of A then*

$$B/I\cap B \cong B+I/I.$$

2 Modules

Definition 2.1. Let R be a ring. A *left R-module* is an abelian group $(M, +)$ and a product $R \times M \to M$ satisfying

- $1a = a$
- $(r_1 r_2)a = r_1(r_2 a)$
- $(r_1 + r_2)a = r_1 a + r_2 a$
- $r(a + b) = ra + rb$

Some examples:
Example 2.2. If R happens to be a field then a left R-module is a vector over R.
Example 2.3. If $M \subseteq R$ then M is a a left R-module if and only if M is a left ideal.
Example 2.4. Abelian groups are $\mathbb{Z}$ modules by defining

$$ng = \begin{cases} \sum_{i=1}^{n} g & \text{if } n > 0 \\ \sum_{i=1}^{n} (\text{-}g) & \text{if } n < 0 \\ 0 & \text{if } n = 0. \end{cases}$$

Example 2.5. If R is a subring of S then S is a left R-module by using multiplication in S.

We can see that R-modules are algebraic structures in the view of universal algebra. Take $(M, +, -0, \{m_r\}_{r \in R})$ together with the axioms for Abelian groups and the four axioms in Definition 2.1. This gives us the definition for *submodule* and module *homomorphism*, *isomorphism*, *epimorphism* and *homomorphism*. We also know that the three isomorphism theorems hold.

Note: We shall write $N \leq M$ to mean that an R-module, N, is a submodule of an R-module, M.

The three isomorphism theorems for modules are:
Theorem 2.6 (1^{st} isomorphism theorem for modules)**.** *Let $f : M \to N$ be a homomorphism of R-modules then there exists an injective map $g : {}^{M}/_{Ker(f)} \to N$ and if f is onto then g is onto.*
Theorem 2.7 (2^{nd} isomorphism theorem for modules)**.** *Let A, B, C be R-modules with $C \leq\leq B \leq A$ then*

$${}^{(A/C)}/_{(B/C)} \cong {}^{A}/_{B}.$$

Theorem 2.8 (3^{rd} isomorphism theorem for modules)**.** *Let A, B, C be R-modules with $B \leq A$ and $C \leq A$ then*

$${}^{B}/_{B \cap C} \cong {}^{B+C}/_{C}.$$

If M_1 and M_2 are R-modules then so are $M_1 + M_2$ and $M_1 \cap M_2$.

If $\{M_i\}_{i \in I}$ is a chain of submodules of M then $\cup_{i \in I} M_i$ is a submodule of M.
Definition 2.9. Let $\{M_i\}_{i \in I}$ be a collection of submodules of M. Define $\sum_{i \in I} M_i$ to be the set of all finite sums of elements of the M_i.

We can see that $\sum_{i \in I} M_i$ is a submodule of M and also that $\sum_{i \in I} M_i$ is the smallest submodule of M containing all every M_i.
Definition 2.10. Let R be a ring and M an R-module. For any $a \in M$ there is an R-module homomorphism:

$$\begin{array}{rccc} f_a : & R & \to & M \\ & r & \to & ra \end{array}$$

3 Exact sequences and commutative diagrams

Definition 3.1. A sequence of homomorphism of modules:

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$$

is *exact* at B if $f(A) = \text{Ker}(g)$.

A sequence is *exact* if it is exact at each intermediate module.
Example 3.2.

$$O \to A \xrightarrow{f} B$$

is exact if and only if $0 = \text{Ker}(f)$, meaning f is a monomorphism.
Example 3.3.

$$A \xrightarrow{f} B \to 0$$

is exact if and only if $f(A) = B$, meaning f is an epimorphism.
Example 3.4.

$$0 \to A \xrightarrow{f}\to B \to 0$$

is exact if and only if it is exact at A and exact at B, meaning f is an isomorphism.
Definition 3.5. A exact sequence of the form:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called a *short exact sequence.*

For a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

- exactness at A means that f is a monomorphism,
- exactness at C means that g is an epimorphism and
- exactness at B means that $f(A) = \text{Ker}(g)$.

This means by the 1^{st} isomorphism theorem that $B/f(A) = B/\text{Ker}(g) = C$.
Example 3.6. Let X, V, W be vector spaces and

$$0 \to V \xrightarrow{f} W \xrightarrow{g} X \to 0$$

be exact. Let A be the matrix of f ($f(v) = Av$) and Let B be the matrix of g (($g(w) = Aw$).

Then $\text{Col}(A) = \text{Nul}(B)$. Since $\dim(\text{Col(A)}) = \text{rank}(A) = \dim(f(V)) = \dim(V)$ and $\dim(\text{Nul}(B)) = \text{nullity}(B)$, we have $\text{nullity}(B) = \dim(V)$. By the rank-nullity theorem we have that $\text{nullity}(B) + \text{rank}(B) = \dim(W)$. Since $frank(B) = \dim(\text{Col}(B)) = \dim(g(W)) = \dim(X)$ we have that

$$\dim(W) = \dim(V) + \dim(X).$$

Now we will look at commutative diagrams which are a kind of arrow picture.

A *commutative diagram* is a digraph with a module at each vertex, a homomorphism compatible with each of its endpoints' modules for each directed edge and for any two paths between from vertex M to vertex N, the composition of homomorphism along each path give the same homomorphism.

Example 3.7. Consider the diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow[j]{} & V \end{array}$$

If the diagram above commutes then $g \circ f = j \circ h$.

Example 3.8. Another example of a commutative diagram is the diagram in the 1^{st} isomorphism theorem.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle \nu}\searrow \quad \nearrow{\scriptstyle g} & \\ & A/\mathrm{Ker}(f) & \end{array}$$