MATH 817 ASSIGNMENT 4 SOLUTIONS

(1) (a) Let
$$c = \; A \xrightarrow{\alpha} C \xleftarrow{\beta} B$$
be an object of $\mathcal{C}'$. Then define id_c to be id_C in $\mathcal{C}$, which is a valid morphism of $\mathcal{C}'$ since
$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & C & \xleftarrow{\beta} & B \\ \big\downarrow{\scriptstyle \mathrm{id}_A} & & \big\downarrow{\scriptstyle \mathrm{id}_C} & & \big\downarrow{\scriptstyle \mathrm{id}_B} \\ A & \xrightarrow{\alpha} & C & \xleftarrow{\beta} & B \end{array}$$
commutes. If θ and ϕ are morphisms of $\mathcal{C}'$ such that
$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & C & \xleftarrow{\beta} & B \\ \big\downarrow{\scriptstyle \mathrm{id}_A} & & \big\downarrow{\scriptstyle \theta} & & \big\downarrow{\scriptstyle \mathrm{id}_B} \\ A & \xrightarrow{\alpha'} & C' & \xleftarrow{\beta'} & B \end{array}$$
and
$$\begin{array}{ccccc} A & \xrightarrow{\alpha'} & C' & \xleftarrow{\beta'} & B \\ \big\downarrow{\scriptstyle \mathrm{id}_A} & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \mathrm{id}_B} \\ A & \xrightarrow{\alpha''} & C'' & \xleftarrow{\beta''} & B \end{array}$$
commute then so does
$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & C & \xleftarrow{\beta} & B \\ \big\downarrow{\scriptstyle \mathrm{id}_A} & & \big\downarrow{\scriptstyle \theta} & & \big\downarrow{\scriptstyle \mathrm{id}_B} \\ A & \xrightarrow{\alpha'} & C' & \xleftarrow{\beta'} & B \\ \big\downarrow{\scriptstyle \mathrm{id}_A} & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \mathrm{id}_B} \\ A & \xrightarrow{\alpha''} & C'' & \xleftarrow{\beta''} & B \end{array}$$
Thus $\phi \circ \theta$ is a morphism of $\mathcal{C}'$.
Composition in $\mathcal{C}'$ is associative since composition in $\mathcal{C}$ is associative so all three columns of the commutative diagram for the composition in $\mathcal{C}'$ can be taken in either order. Also $\mathrm{id}_{C'} \circ \theta = \theta \circ \mathrm{id}_C$ so the middle column of the commutative diagram for each identity composition is simply θ, and the outer columns are always just id_A and id_B so the identity axiom holds in $\mathcal{C}'$. Make the hom sets of $\mathcal{C}'$ disjoint by fiat (that is a morphism of $\mathcal{C}'$ is a morphism of $\mathcal{C}$ satisfying the required commutativity along with the information of the domain and range objects.)

(b) An initial object in $\mathcal{C}'$ is a diagram

$$A \xrightarrow{\alpha} I \xleftarrow{\beta} B$$

such that for every other object

$$A \xrightarrow{\alpha'} C' \xleftarrow{\beta'} B$$

there is a unique morphism

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_A"] & I \arrow[d, "\theta"] & B \arrow[l, "\beta"'] \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "\alpha'"] & C' & B \arrow[l, "\beta'"'] \end{tikzcd}$$

contracing the id edges we get exactly the diagram for the coproduct of A and B in $\mathcal{C}$.

(c) Let A and B again be objects in a category $\mathcal{C}$. Define a new category $\mathcal{C}''$ whose objects are diagrams

$$A \xleftarrow{p} C \xrightarrow{q} B$$

where C is an object in $\mathcal{C}$ and p and q are morphisms in $\mathcal{C}$. Define a morphism in $\mathcal{C}''$ to be a morphism θ in $\mathcal{C}$ that makes the following diagram commute

$$\begin{tikzcd} A \arrow[d, "\mathrm{id}_A"] & C \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\theta"] & B \arrow[d, "\mathrm{id}_B"] \\ A & C' \arrow[l, "p'"'] \arrow[r, "q'"] & B \end{tikzcd}$$

Define composition by stacking squares and let the identity morphism of $\mathcal{C}''$ come from the identity morphism in the middle column as in the previous construction. Then the composition and identity morphisms are again inherited from the same properties of $\mathcal{C}$ and the hom sets can again be made disjoint by fiat. A terminal object of $\mathcal{C}''$ is a diagram

$$A \xleftarrow{p} T \xrightarrow{q} B$$

such that for every other object

$$A \xleftarrow{p'} C' \xrightarrow{q'} B$$

there is a unique morphism

$$\begin{tikzcd} A \arrow[d, "\mathrm{id}_A"] & C' \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[d, "\theta"] & B \arrow[d, "\mathrm{id}_B"] \\ A & T \arrow[l, "p"'] \arrow[r, "q"] & B \end{tikzcd}$$

contracing the id edges we get exactly the diagram for the product of A and B in $\mathcal{C}$.

(2) Let A, B, C be abelian groups with homomorphisms $\phi : A \to C$, $\psi : B \to C$. Let $P = \{(a, b) \in A \times B | \phi(a) = \psi(b)\}$. Then P is an abelian group as $\phi(1) = 1 = \psi(1)$ so $(1, 1) \in P$; $\phi(a) = \psi(b)$ and $\phi(a') = \psi(b')$ implies $\phi(aa') = \psi(bb')$ so P is closed under multiplication; and $\phi(a) = \psi(b)$ implies $\phi(a^{-1}) = \psi(b^{-1})$ so P is closed under

inverses. Let $\alpha : P \to A$ be projection onto the first coordinate and let $\beta : P \to B$ be projection onto the second coordinate, then

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & A \\ \Big\downarrow \beta & & \Big\downarrow \phi \\ B & \xrightarrow{\psi} & C \end{array}$$

commutes. If

$$\begin{array}{ccc} D & \xrightarrow{\alpha'} & A \\ \Big\downarrow \beta' & & \Big\downarrow \phi \\ B & \xrightarrow{\psi} & C \end{array} \tag{1}$$

also commutes then we can construct $\theta : D \to P$ by $\theta(d) = (\alpha'(d), \beta'(d))$. This does land in P because (1) commutes so $\phi(\alpha'(d)) = \psi(\beta'(d))$, and it satisfies the required commutativity because $\alpha(\theta(d)) = \alpha'(d)$ and likewise for β. Finally this morphism is unique because if $\eta : D \to P$ is also such a morphism then for all $d \in D$, $\alpha(\eta(d)) = \alpha'(d) = \alpha(\theta(d))$ and likewise $\beta(\eta(d)) = \beta(\theta(d))$ which implies that $\eta(d) = \theta(d)$. Therefore P is the pullback.

(3) Let $T(G) = G/G'$ for any group G. For a group homomorphism let $T(\phi : G \to H) = T(\phi) : G/G' \to H/H'$ where $T(\phi)(G'x) = H'\phi(x)$. $T(\phi)$ is well defined since if $G'x = G'y$ then $xy^{-1} \in G'$ so $xy^{-1} = [a, b]$ so $\phi(x)\phi(y)^{-1} = \phi(xy^{-1}) = [\phi(a), \phi(b)]$ which tells us that $H'\phi(x) = H'\phi(y)$. $T(\phi)$ is a homomorphism since $T(\phi)(G'xy) = H'\phi(xy) = H'\phi(x)\phi(y)$. T respects compositions since if also $\psi : H \to K$, then $T(\psi\circ\phi)(G'x) = K'(\psi\circ\phi)(x)$ while $T(\psi)\circ T(\phi)(G'x) = T(\psi)(H'\phi(x)) = K'(\psi\circ\phi)(x)$. T respects identity morphisms since $T(\mathrm{id}_G)(x) = G'x = \mathrm{id}_{G/G'}(G'x)$. Thus T is a functor.

(4) Let U be the forgetful functor from groups to sets. We need a natural transformation τ such that

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{Z}, G) & \xrightarrow{\tau_G} & UG \\ \Big\downarrow f_* & & \Big\downarrow Uf \\ \mathrm{Hom}(\mathbb{Z}, H) & \xrightarrow{\tau_H} & UH \end{array}$$

for any group homomorphism $f : G \to H$, and such that each τ_G is a set bijection. Note that any $h \in \mathrm{Hom}(\mathbb{Z}, G)$ is determined by where it sends 1, and that since $\mathbb{Z}$ is freely generated by 1, any value for $h(1)$ is valid. So define $\tau_G : \mathrm{Hom}(\mathbb{Z}, G) \to UG$ by $\tau_G(h) = h(1)$. This is a set bijection by the above observations. The required diagram commutes because $f(\tau_G(h)) = f(h(1)) = f_*(h)(1) = \tau_H(f_*(h))$. Thus the forgetful functor from groups to sets is represented by $\mathbb{Z}$.

(5) Let $D_8 = \langle a, b | a^4 = b^2 = 1, bab = a^3\rangle$ and let $Z_2 = \langle c | c^2 = 1\rangle$. Write $G = \langle a, b, c | a^4 = b^2 = c^2 = 1, bab = a^3, ac = ca, bc = cb\rangle$. Let $a^x = a^2$, $b^x = c$ and $c^x = b$. To check that this action of x defines an endomorphism of G it suffices to check that all the

relations defining G are in its kernel. Check

$$
\begin{aligned}
&(a^x)^4 = a^8 = 1 \\
&(b^x)^2 = c^2 = 1 \\
&(c^x)^2 = b^2 = 1 \\
&b^x a^x b^x a^x = ca^2ca^2 = 1 \\
&(a^x)^{-1}(c^x)^{-1}a^x c^x = a^{-2}b^{-1}a^2b = 1 \\
&(b^x)^{-1}(c^x)^{-1}b^x c^x = c^{-1}b^{-1}cb = 1
\end{aligned}
$$

Now $Z(G) = \{1, a^2, c, a^2c\}$, but $c^x = b$, so $Z(G)$ is not an X-subgroup.