# MATH 821, SPRING 2013, ASSIGNMENT 4 SOLUTIONS

(1) Given a self conjugate partition $\lambda$ of $n$. Consider the Ferrers diagram of $\lambda$. Take those boxes which are in the top row or the first colum (including the top corner which is in both). Make those boxes the first row of the Ferrers diagram of a new partition. Note that the remaining boxes from the Ferres diagram of $\lambda$ still form a self conjugate partition. Take the top row and first column of what remains. Make these boxes the second row of the new partition. Continue likewise. Let $\mu$ be the partition whose Ferrers diagram is the one just created.

Any self conjugate partition has the same number of boxes in the top row as in the first column, and exactly one box is in both. Thus the number of boxes in either the first row or the first column is odd. Thus $\mu$ has only odd parts. In any partition the second row has at most as many elements as the first row and the second column has at most as many elements as the first column. Assume the second row and column are both nonempty. After removing the first row and column, either exactly one box has been removed from the next row and column. Thus consecutive parts of $\mu$ must differ by at least 2. Therefore $\mu$ is a partition with odd and distinct parts.

For the inverse map, let $\mu$ be a partition with odd and distinct parts. Consider the Ferrers diagram of $\mu$. For row $i$ of the Ferrers diagram of $\mu$, remove the boxes, place one at position $(i, -i)$ (counting like cartesian coordinates) of a new Ferrers diagram. Place half of the remaining boxes in a row after this first box and place the other half in a column below this first box. This map is the inverse of the above map, and hence we have a bijection.

(2) Let $G$ be a connected QED Feynman graph. Let $e_i^p$ be the number of internal photon edges of $G$, $e_i^f$ the number of internal fermion edges of $G$, and $e_x^p$ and $e_x^f$ the corresponding number of external edges of each type. Let $v$ be the number of vertices of $G$. Let $\ell$ be the number of independent cycles of $G$. Then by Euler's formula

$$v - e_i^p - e_i^f + \ell = 1$$

but by 3-regularity (each vertex has a photon as well as a fermion in and out)

$$3v = 2e_i^p + e_x^p + 2e_i^f + e_x^f$$

and

$$2(2e_i^p + e_x^p) = 2e_i^f + e_x^f$$

so

$$3v = 3(2e_i^p + e_x^p)$$

Using the above we have that the superficial degree of divergence of $G$ is

$$\begin{aligned} 4\ell - 2e_i^p - e_i^f &= 4 + 2e_i^p + 3e_i^f - 4v \\ &= 4 + 8e_i^p + 3e_x^p - \frac{3}{2}e_x^f - 4v \\ &= 4 - e_x^p - \frac{3}{2}e_x^f \end{aligned}$$

which depends only on the external edges. Therefore QED is renormalizable. For disconnected graphs it will also depend on the number of components which is fine.

(3) Let us take each path going up to $\infty$ – this is the same issue as with the number of variable, we can cut off at any hight which is high enough to not miss any horizontal steps.

Since the rows of the tableau are weakly increasing the path uniquely determines the row as follows: read off the height of each horizontal step, these will appear in weakly increasing order as we follow the path from bottom to top, these fill the boxes of the row. The number of boxes in the row is the horizonal offset between the top and bottom of the path, since it is also the number of horizontal steps. Thus sets of paths beginning and ending at the same points will yield the same Ferrers shape.

Suppose paths $i$ and $i+1$ meet at a common vertex at height $k$. Since path $i+1$ started one point to the left of path $i$, path $i+1$ must have had one more horizontal step than path $i$ by height $k$. Thus row $i+1$ of the tableau has at least one more box containing an entry $\leq k$ than row $i$ does. This contradicts column strictness. Therefore the paths are disjoint.

Conversely consider any two paths built of up and east steps and starting one unit apart. If these paths are disjoint then at any given height $k$, the leftward path has no more horizontal steps at or below height $k$ than the rightwards path does. Thus viewing these as rows of filled boxes, the row corresponding to the leftwards path has strictly fewer entries $\leq k$ than the row corresponding to the rightwards path does. Thus putting the row of the rightwards path on top of the other row, we see that the columns are strictly increasing down rows and that the top row is of the same length or longer as the lower row.

(4) *Answers will vary*